EIGENVALUES OF THE LAPLACE OPERATOR ON CERTAIN MANIFOLDS

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To every compact Riemannian manifold M there corresponds the sequence $0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ of eigenvalues for the Laplace operator on M. It is not known just how much information about M can be extracted from this sequence.[1] This note will show that the sequence does not characterize M completely, by exhibiting two 16-dimensional toruses which are distinct as Riemannian manifolds but have the same sequence of eigenvalues.

By a *flat torus* is meant a Riemannian quotient manifold of the form R^n/L, where L is a lattice (= discrete additive subgroup) of rank n. Let L^* denote the dual lattice, consisting of all $y \in R^n$ such that $x \cdot y$ is an integer for all $x \in L$. Then each $y \in L^*$ determines an eigenfunction $f(x) = \exp(2\pi\, i x \cdot y)$ for the Laplace operator on R^n/L. The corresponding eigenvalue λ is equal to $(2\pi)^2 y \cdot y$. Hence, the number of eigenvalues less than or equal to $(2\pi r)^2$ is equal to the number of points of L^* lying within a ball of radius r about the origin.

According to Witt[2] there exist two self-dual lattices $L_1, L_2 \subset R^{16}$ which are distinct, in the sense that no rotation of R^{16} carries L_1 to L_2, such that each ball about the origin contains exactly as many points of L_1 as of L_2. It follows that the Riemannian manifolds R^{16}/L_1 and R^{16}/L_2 are not isometric, but do have the same sequence of eigenvalues.

In an attempt to distinguish R^{16}/L_1 from R^{16}/L_2 one might consider the eigenvalues of the Hodge-Laplace operator $\Delta = d\delta + \delta d$, applied to the space of differential p-forms. However, both manifolds are flat and parallelizable, so the identity

$$\Delta(f\, dx_{i_1} \wedge \ldots \wedge dx_{i_p}) = (\Delta f) dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

shows that one obtains simply the old eigenvalues, each repeated $\binom{16}{p}$ times.

[1] Compare Avakumović, V., "Über die Eigenfunktionen auf geschlossenen Riemannschen Mannigfaltigkeiten," *Math. Zeits.*, **65**, 327–344 (1956).

[2] Witt, E., "Eine Identität zwischen Modulformen zweiten Grades," *Abh. Math. Sem. Univ. Hamburg*, **14**, 323–337 (1941). See p. 324. I am indebted to K. Ramanathan for pointing out this reference.