SYSTOLE INEQUALITIES FOR ARITHMETIC LOCALLY SYMMETRIC SPACES

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Abstract. In this paper we study the systole growth of arithmetic locally symmetric spaces up congruence covers and show that this growth is at least logarithmic in volume. This generalizes previous work of Buser and Sarnak, and Katz, Schaps and Vishne in the context of compact arithmetic hyperbolic manifolds of dimension 2 and 3.

1. Introduction

The *systole* of a compact Riemannian manifold M is the least length of a non-contractible loop on M. The systole $\mathrm{sys}(M)$ of M and the volume $\mathrm{Vol}(M)$ of M are deeply related and have been the focus of considerable research. For instance, Gromov [10] showed that for a closed aspherical n-manifold M, there exists a constant $c := c(n)$ such that

$$\mathrm{sys}(M) \leq c(\mathrm{Vol}(M))^{1/n}. \tag{1}$$

Note that all compact locally symmetric spaces of nonpositive curvature are closed aspherical.

Of particular interest has been the study of how systoles grow along congruence covers of a given base manifold M. Buser and Sarnak [7] showed that when M is a compact arithmetic hyperbolic surface arising from a quaternion division algebra over $\mathbf{Q}$ there exists a constant $c := c(M)$ such that the principal congruence covers $\{M_I\}$ of M satisfy

$$\mathrm{sys}(M_I) \geq \frac{4}{3}\log(g(M_I)) - c, \tag{2}$$

where $g(M_I)$ denotes the genus of M_I and log denotes the natural logarithm. This result was later extended to principal congruence covers of arbitrary compact arithmetic hyperbolic surfaces by Katz, Schaps and Vishne [14]. Furthermore, Katz, Schaps and Vishne proved an analogous result for compact arithmetic hyperbolic 3-manifolds; namely, for a suitable constant $c := c(M)$, the principal congruence covers $\{M_I\}$ of M satisfy

$$\mathrm{sys}(M_I) \geq \frac{2}{3}\log(\|M_I\|) - c, \tag{3}$$

where $\|M_I\|$ denotes the simplicial volume of M_I (see [29, Chapter 6]). These results were later generalized by Murillo to arithmetic hyperbolic manifolds of dimension n which are of the first type [22] and to Hilbert modular varieties [21] in the case that I varies across the set of prime ideals in a certain number field.

The goal of this paper is to prove a generalization of the aforementioned Buser–Sarnak inequality (2) for all arithmetic simple locally symmetric manifolds.

Unlike in the case of hyperbolic manifolds, there are multiple natural choices for how to scale the metric on a generic locally symmetric manifold. In Section 2 we discuss such choices and explain in detail how scaling the metric affects the systole, volume, and systole growth up a tower of covers. In particular, in Proposition 2.2 we show that when systole growth is at least logarithmic up a tower, then this fact is independent of the scaling of the metric.

Before proving the general case, we focus on *standard special linear manifolds*. A *special linear manifold* of degree n is a manifold of the form $M_\Gamma := \Gamma \backslash \mathrm{SL}_n(\mathbf{R}) / \mathrm{SO}(n)$ where $\Gamma \subset \mathrm{SL}_n(\mathbf{R})$ is a torsion-free lattice. We call such lattices *standard* when they arise from central simple algebras (see Section 4), and we note that this terminology is in analogy to how arithmetic hyperbolic manifolds arising from quadratic forms are called *standard arithmetic hyperbolic manifolds* [20, 4.10]. Note that standard special linear manifolds are arithmetic and all arithmetic hyperbolic 2-manifolds are also standard special linear manifolds of degree 2.

With Proposition 2.2 in mind, we choose a convenient normalization of the metric on special linear manifolds so that the sectional curvature is bounded between 0 and 1, and we call this normalization the *geometric metric* g. Given a standard special linear manifold M and a rational prime power p, we denote by $\{M_{p^m}\}$ the principal p-congruence tower of M (see Section 5). We show that the systole growth up all but finitely many p-congruence towers is at least logarithmic in volume.

Theorem A. *Let M be a standard special linear manifold of degree n, $n \geq 2$. There exists a constant $c := c(M, g)$ such that for all but finitely many primes p and all positive integers m,*

$$\operatorname{sys}(M_{p^m}, g) \geq \frac{2\sqrt{2}}{n(n^2-1)} \log(\operatorname{Vol}(M_{p^m}, g)) - c. \tag{4}$$

In the special case when $n = 2$ and M is compact, the Gauss–Bonnet theorem states $g(M_{p^m}) = \frac{\operatorname{Vol}(M_{p^m}, g)}{2\pi} + 1$, and hence Theorem A gives

$$\operatorname{sys}(M_{p^m}) \geq \frac{\sqrt{2}}{3} \log(g(M_{p^m})) - c'$$

where $c' = c'(M)$ is a constant. Observe that this is close to recovering (2).

These methods enable use to prove a similar result for noncompact standard arithmetic manifolds. Note that all even dimensional arithmetic hyperbolic manifolds are standard. Unless otherwise stated, hyperbolic manifolds will be given the hyperbolic metric in which they have constant sectional curvature -1, and the systole and volume will be scaled accordingly.

Theorem B. *Let N be an noncompact standard arithmetic hyperbolic n-manifold, $n \geq 2$. There exists a constant $c := c(N)$ such that for all but finitely many primes p and all positive integers m,*

$$\operatorname{sys}(N_{p^m}) \geq \frac{2\sqrt{2}}{n(n+1)^2} \log(\operatorname{Vol}(N_{p^m})) - c. \tag{5}$$

A major ingredient in our proofs of Theorems A and B is our Trace-Length Bounds Theorem 3.1. In both $\mathrm{SL}_2(\mathbf{R})$ and $\mathrm{SL}_2(\mathbf{C})$, the translation length of a semisimple element can be understood in terms of the trace of the element. This relationship has proven to be extremely useful, as the trace is well understood from a number theoretic perspective. In $\mathrm{SL}_n(\mathbf{R})$, $n \geq 3$, the relationship between translation length and trace is more nuanced. Nevertheless, in our Trace-Length Bounds Theorem 3.1 we prove upper and lower bounds for the translation length of a semisimple element in terms of the element's trace.

In Section 7 we show that each arithmetic simple locally symmetric manifold N is commensurable to an immersed totally geodesic submanifold of a standard special linear manifold of explicitly bounded degree (Theorem 7.1). Relative to this immersion, we endow each

N with the subspace metric which we also denote g. For each rational prime p, this immersion induces a p-congruence tower $\{N_{p^m}\}$ above N. While the induced p-congruence tower seems dependent upon the immersion, it is in fact natural in that it is commensurable of bounded distance (see Section 2) to the tower associated to the principal congruence subgroups $\ker(\mathrm{G}(\mathcal{O}_k) \to \mathrm{G}(\mathcal{O}_k/p^m\mathcal{O}_k))$ (see Remark 7.3).

In addition to its associated Riemannian volume, each N has an *arithmetic measure* μ_a in the sense of Prasad [25]. We believe that μ_a is the most natural measure for a general N, the most easily computable thanks to Prasad's volume formula [25, Theorem 3.7], and hence that stating our results in terms of μ_a is most likely to be of use. That being said, there is an analogous statement for when arithmetic measure is replaced by metric volume.

Theorem C. *Let N be an arithmetic simple locally symmetric manifold of dimension n and arithmetic measure $\mu_a(N) < v$. Then for all but finitely many primes p and all positive integers m,*

$$\mathrm{sys}(N_{p^m}, g) \geq c_1 \log(\mu_a(N_{p^m})) - c_2. \tag{6}$$

Where $c_1 := c_1(n, v)$ and $c_2 := c_2(N)$ are explicit constants.

All simple locally symmetric manifolds that are not either real or complex hyperbolic are arithmetic, and hence Theorem C applies. In the case of arithmetic hyperbolic manifolds with the hyperbolic metric, we then prove the following theorem in which we explicitly determine the dependence of the multiplicative constant on the volume.

Theorem D. *Let N be an arithmetic hyperbolic n-manifold with hyperbolic volume less than V. There exists an absolute, effectively computable constant $c_1 := c_1(n) > 0$, and a constant $c_2 := c_2(N)$ such that for all but finitely many primes p and all positive integers m,*

$$\mathrm{sys}(N_{p^m}) \geq \frac{c_1}{(\log(V))^3} \log(\mathrm{Vol}(N_{p^m})) - c_2. \tag{7}$$

In Section 9, we show how to explicitly compute the constants c_1 above. Observe that the multiplicative constants in Theorems A and B depend only on dimension, while the multiplicative constants in Theorems C and D depend on dimension and volume.

Remark. All of the results in this paper hold as well for simple locally symmetric *orbifolds*. Note that in the context of locally symmetric orbifolds a closed geodesic is not defined to be locally length minimizing but rather to be a closed curve which lifts to a geodesic in the universal cover.

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2. Preliminaries on Metrics, Lengths, Volumes, and Towers

In this paper we assume some familiarity with Riemannian manifolds, Lie groups, Lie algebras, and symmetric spaces. For a detailed reference on these topics, we refer the reader to [12]. We now record a few facts and establish some terminology that we use throughout the paper.

Let (M, g) be a finite volume Riemannian manifold and $c_1, c_2 \in \mathbf{R}$ be such that

$$\mathrm{sys}(M, g) \geq c_1 \log(\mathrm{Vol}(M, g)) - c_2. \tag{8}$$

Such a systole-volume bound behaves nicely when scaling the metric or lifting to covers:

Lemma 2.1.

(i) *If* $\alpha \in \mathbf{R}_{>0}$*, then* $\operatorname{sys}(M, \alpha g) \geq c_1' \log(\operatorname{Vol}(M, \alpha g)) - c_2'$ *where*

$$c_1' = \sqrt{\alpha} c_1 \qquad\qquad c_2' = \sqrt{\alpha}\left(c_2 + \frac{c_1 \dim M}{2} \log \alpha\right).$$

(ii) *If* $M_I \to M$ *is an* s*-sheeted cover, then* $\operatorname{sys}(M_I) \geq c_1 \log(\operatorname{Vol}(M_I)) - c_2'$*. where*

$$c_2' = c_2 + c_1 \log s.$$

Proof. Scaling the metric scales the corresponding systole and volume (see [8, Chapter 1]):

$$\operatorname{sys}(M, \alpha g) = \sqrt{\alpha} \operatorname{sys}(M, g) \qquad\qquad \operatorname{Vol}(M, \alpha g) = \alpha^{\frac{\dim M}{2}} \operatorname{Vol}(M, g). \tag{9}$$

An s-sheeted cover satisfies $\operatorname{Vol}(M_I) = s \operatorname{Vol}(M)$. The results follow by substituting these values into (8). $\square$

A *tower of covers* $\{M_I\}_{I \in \mathcal{S}}$ of M is a set of finite sheeted covers of M indexed by a poset $\mathcal{S}$ such that if $I < J$, then M_J covers M_I. If the systole of each manifold in the tower satisfies a logarithmic volume lower bound as in (8) with the same constants c_1 and c_2, then we say the *systole growth is at least logarithmic in volume* up the tower. In other words, the systole growth up a tower of covers $\{M_I\}_{I \in \mathcal{S}}$ of M is at least logarithmic in volume if there are constants c_1 and c_2 which depend only on M such that $\operatorname{sys}(M_I) \geq c_1 \log(\operatorname{Vol}(M_I)) - c_2$ for all manifolds M_I in the tower.

Two locally symmetric spaces M and M' are *commensurable* if they share a common finite sheeted cover. Equivalently, if Γ and Γ' are their corresponding lattices, then $\Gamma \cap \Gamma'$ has finite index in Γ and Γ'. If $\{M_I\}_{I \in \mathcal{S}}$ is a tower of covers of M, then there is a canonical *induced* tower $\{M_I'\}_{I \in \mathcal{S}}$ of M' such that for each $I \in \mathcal{S}$, M_I' is a finite sheeted cover of M_I of degree $|M_I' : M_I| \leq |\Gamma : \Gamma \cap \Gamma'|$. This is obtained by just intersecting the appropriate lattices.

We say two towers $\{M_I\}_{I \in \mathcal{S}}$ and $\{M_I'\}_{I \in \mathcal{S}}$ are *commensurable*, if there exists a tower $\{M_I''\}_{I \in \mathcal{S}}$ where for each $I \in \mathcal{S}$, M_I'' is a common finited sheeted cover of M_I and M_I'. We say that commensurable towers are of *bounded distance* if there exists an integer $s \geq 1$ such that for each $I \in \mathcal{S}$, the covering maps $M_I'' \to M_I$ and $M_I'' \to M_I'$ are of no more than s sheets. Commensurable of bounded distance is an equivalence relation between towers. Induced towers are of bounded distance.

It follows from Lemma 2.1 (ii) that if the systole growth up $\{M_I\}_{I \in \mathcal{S}}$ is at least logarithmic in volume, and $\{M_I'\}_{I \in \mathcal{S}}$ is commensurable of bounded distance to $\{M_I\}_{I \in \mathcal{S}}$, then the systole growth up $\{M_I'\}_{I \in \mathcal{S}}$ is also at least logarithmic in volume, and furthermore, they have the same multiplicative constant. We record these observations in the proposition below.

Proposition 2.2. *Let* M *be a finite volume locally symmetric space and* $\{M_I\}_{I \in \mathcal{S}}$ *a tower of covers of* M*. If systole growth is at least logarithmic in volume up the tower,*

(i) *Then this property is independent of the scaling of the metric,*
(ii) *The systole growth is at least logarithmic in volume up a commensurable tower of bounded distance, and*
(iii) *Upon fixing the metric, the multiplicative constant* c_1 *is an invariant of the bounded distance commensurability class of* $\{M_I\}_{I \in \mathcal{S}}$*.*

Sometimes it is more convenient or natural to work with the scaled measure $\mu_\beta = \beta \operatorname{Vol}$, $\beta \in \mathbf{R}_{>0}$, on N. For example, for compact hyperbolic n-manifolds, Gromov showed that there exists a constant $\beta := \beta(n)$ such that simplicial volume is β times hyperbolic volume [29, Theorem 6.2]. In Section 9, we shall be considering the arithmetic measure of an arithmetic simple locally symmetric space, which is a scaling of the metric volume. A direct computation shows that that if there exists constants c_1 and c_2 such that $\operatorname{sys}(N) \geq c_1 \log(\operatorname{Vol}(N)) - c_2$, then $\operatorname{sys}(N) \geq c_1 \log(\mu_\beta(N)) - c_2'$ for $c_2' = (c_2 - c_1 \log \beta)$. In such a case, it follows that systole growth up a tower is at least logarithmic in metric volume if and only if systole growth is at least logarithmic in measure (c.f. (3)).

The symmetric space $\operatorname{SL}_n(\mathbf{R})/\operatorname{SO}(n)$ comes naturally equipped with two Riemannian metrics: (1) the *Killing metric* determined by the Killing form $B(X,Y) = 2n \operatorname{tr}(XY)$ on $\mathfrak{sl}_n(\mathbf{R})$ and (2) the *geometric metric*, in which the hyperbolic slices corresponding to the natural inclusions $\operatorname{SL}_2(\mathbf{R}) \to \operatorname{SL}_n(\mathbf{R})$ have constant sectional curvature of -1. These metrics are constant multiples of one another. Relative to the scaled Killing form αB, $\alpha \in \mathbf{R}_{>0}$, the curvature of a section determined by orthonormal vectors $X, Y \in \mathfrak{sl}_n(\mathbf{R})$ is $K(X,Y) = \frac{2n}{\alpha} \operatorname{tr}([X,Y]^2)$ [12, V.3.1]. It follows that the geometric metric is determined by

$$\langle X, Y\rangle := \frac{1}{n} B(X,Y) = 2\operatorname{tr}(XY), \qquad X, Y \in \mathfrak{sl}_n(\mathbf{R}). \tag{10}$$

Many papers on locally symmetric spaces use the Killing metric, such as [26], however since we are interested in isometric immersions of locally symmetric spaces of smaller dimensions, we normalize to the geometric metric.

Each $x \in \operatorname{SL}_n(\mathbf{R})$ has Jordan decomposition $x = x_s x_u$ where x_s is *semisimple* and x_u is *unipotent*. Semisimple elements in $\operatorname{SL}_n(\mathbf{R})$ are diagonalizable, possibly over $\mathbf{C}$. When Γ is a cocompact lattice in $\operatorname{SL}_n(\mathbf{R})$, the Godement Compactness Criterion implies that it only has semisimple elements [32, Theorem 5.3.3]. Every semisimple element has a polar decomposition $x = x_h x_e$ where its *hyperbolic part* x_h has all positive real eigenvalues and its *elliptic part* x_e has eigenvalues that lie on the unit circle. In particular, if $\{a_1, \ldots, a_n\}$ are the eigenvalues of x, then $\{|a_1|, \ldots, |a_n|\}$ are the eigenvalues of x_h. Unless stated otherwise, in what follows $x \in \operatorname{SL}_n(\mathbf{R})$ will denote a semisimple element and $\{a_1, \ldots a_n\}$ its eigenvalues.

Each x stabilizes and translates along a geodesic axis in $\operatorname{SL}_n(\mathbf{R})/\operatorname{SO}(n)$. Let $\ell(x)$ denote the *translation length* of x relative to the geometric metric on $\operatorname{SL}_n(\mathbf{R})/\operatorname{SO}(n)$. Closed geodesics in $\Gamma\backslash \operatorname{SL}_n(\mathbf{R})/\operatorname{SO}(n)$ are in bijective correspondence with Γ-conjugacy classes of semisimple elements in Γ. The length of a closed geodesic associated to the class of x is the translation length of x.

Let $\mathrm{A} \subset \operatorname{SL}_n(\mathbf{R})$ denote the Lie subgroup of diagonal matrices with positive entries and let $\mathfrak{a}$ denote its Lie algebra. The map $\log : \mathrm{A} \to \mathfrak{a} \subset \mathfrak{sl}_n(\mathbf{R})$, sending $(b_1, \ldots, b_n) \mapsto (\log(b_1), \ldots, \log(b_n))$ is an isomorphism. Then $y = \operatorname{diag}(|a_1|, \ldots, |a_n|) \in \mathrm{A}$ is $\operatorname{SL}_n(\mathbf{R})$-conjugate to x_h. Let $Y = \log(y)$. Using (10), (c.f. [16, Section 12.1], [26, Prop. 8.5])

$$\ell(x) = \ell(x_h) = \ell(y) = \sqrt{\langle Y, Y\rangle} = \sqrt{2\operatorname{tr}(Y^2)} = \sqrt{2\sum_{i=1}^{n} (\log|a_i|)^2}. \tag{11}$$

3. Trace-Length Bounds Theorem

In this section we state and prove a fundamental relationship between the traces and translation lengths of elements $x \in \mathrm{SL}_n(\mathbf{R})$. These relationships are particularly valuable since they enable us to leverage number theoretic techniques to analyze traces, thereby giving us geometric data about lengths. The proof of the theorem uses a variety of analytic techniques.

Theorem 3.1 (Trace–Length Bounds)**.** *For $x \in \mathrm{SL}_n(\mathbf{R})$ semisimple,*

$$\sqrt{2}\operatorname{arccosh}\left(\frac{\operatorname{tr}(x_h)}{n}\right) \leq \ell(x) \leq \sqrt{2n}\operatorname{arccosh}\left(\left(\frac{\operatorname{tr}(x_h)}{n}\right)^{n-1}\right). \tag{12}$$

Furthermore, if $|\operatorname{tr}(x)| > n$, then

$$\sqrt{2}\operatorname{arccosh}\left(\frac{|\operatorname{tr}(x)|}{n}\right) \leq \ell(x) \leq \sqrt{2n}\operatorname{arccosh}\left(\left(2 + 2\sum_{l=1}^{n} |\operatorname{tr}(x^l)|\right)^{n-1}\right). \tag{13}$$

Remark 3.2. When $n = 2$, it is known that for x hyperbolic with eigenvalues a and $\frac{1}{a}$,

$$\ell(x) = 2\log|a| = 2\operatorname{arccosh}\left(\frac{|\operatorname{tr}(x)|}{2}\right). \tag{14}$$

Observe that (12) gives a tight upper bound. For $n \geq 3$, no such direct equality is known.

The remainder of this section is dedicated to proving Theorem 3.1. The lower bounds follow from (11) and Proposition 3.4 (below), while Propositions 3.5 and 3.6 (below) along with (11) yield the upper bounds. Recall that for $x \in \mathrm{SL}_n(\mathbf{R})$ semisimple, its eigenvalues $a_1, \ldots a_n$ are complex numbers that satisfy $\sum_{i=1}^n a_i = \operatorname{tr} x$ and $\prod_{i=1}^n a_i = 1$.

Lemma 3.3. *For any $\{a_i\}_{i=1}^n \subset \mathbf{C}$, if $\displaystyle\prod_{i=1}^n a_i = 1$, then $\displaystyle\sum_{i=1}^n |a_i| \geq n$ and $\displaystyle\sum_{i=1}^n |a_i|^{-1} \geq n$.*

Proof. This is simply an application of Jensen's Inequality:

$$\frac{1}{n}\sum_{i=1}^n |a_i| \geq (\prod_{i=1}^n |a_i|)^{\frac{1}{n}} = 1 \text{ and } \frac{1}{n}\sum_{i=1}^n |a_i^{-1}| \geq (\prod_{i=1}^n |a_i|^{-1})^{\frac{1}{n}} = 1.$$

□

Proposition 3.4. *For any $\{a_i\}_{i=1}^n \subset \mathbf{C}$ satisfying $\displaystyle\prod_{i=1}^n a_i = 1$, then:*

$$\operatorname{arccosh}\left(\frac{1}{n}\sum_{i=1}^n |a_i|\right) \leq \sqrt{\sum_{i=1}^n (\log|a_i|)^2}.$$

Furthermore, if $|\sum_i a_i| \geq n$, then:

$$\operatorname{arccosh}\left(\frac{1}{n}\left|\sum_{i=1}^n a_i\right|\right) \leq \operatorname{arccosh}\left(\frac{1}{n}\sum_{i=1}^n |a_i|\right) \leq \sqrt{\sum_{i=1}^n (\log|a_i|)^2}.$$

Note: In Proposition 3.4, if we replace a_i with a_i^{-1} in the three bounds on the left, we get the identical two bounds on the right.

Proof. We will begin by proving an inequality for the sum inside of arccosh.

$$\begin{aligned}
\frac{1}{n}\left|\sum_{i=1}^{n} a_i\right| &\leq \frac{1}{n}\sum_i |a_i| = \frac{2}{n}\left(\sum_i \frac{1}{2}\left(e^{\log|a_i|}+e^{-\log|a_i|}\right) - \frac{1}{2}\sum_i |a_i|^{-1}\right)\\
&= \frac{2}{n}\left(\sum_i \cosh(\log|a_i|) - \frac{1}{2}\sum_i |a_i|^{-1}\right)\\
&= \frac{2}{n}\left(n + \sum_{m=1}^{\infty}\left(\sum_i \frac{(\log|a_i|)^{2m}}{(2m)!}\right) - \frac{1}{2}\sum_i |a_i|^{-1}\right) \quad (15)\\
&\leq \frac{2}{n}\left(\frac{n}{2} + \sum_{m=1}^{\infty}\left(\frac{\left(\sum_i (\log|a_i|)^2\right)^m}{(2m)!}\right)\right) \quad (16)\\
&\leq \frac{2}{n}\left(\frac{n}{2}\sum_{m=0}^{\infty}\left(\frac{\left(\sqrt{\sum_i (\log|a_i|)^2}\right)^{2m}}{(2m)!}\right)\right)\\
&= \cosh\left(\sqrt{\sum_{i=1}^{n}(\log|a_i|)^2}\right) \quad (17)
\end{aligned}$$

Note that equations (15) and (17) follow from the Taylor Series expansion of $\cosh(x) = \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!}$, while equation (16) makes use of Lemma 3.3 to get $n - \frac{1}{2}\sum_i |a_i|^{-1} \leq \frac{n}{2}$.

If $|\sum_i a_i| \geq n$, then

$$\operatorname{arccosh}\left(\frac{1}{n}\left|\sum_{i=1}^{n} a_i\right|\right) \leq \operatorname{arccosh}\left(\frac{1}{n}\sum_{i=1}^{n}|a_i|\right) \leq \sqrt{\sum_{i=1}^{n}(\log|a_i|)^2}$$

since $\operatorname{arccosh}(x)$ is increasing on its domain $[1,\infty)$. If $|\sum_i a_i| < n$, the right inequality still holds. $\square$

Proposition 3.5. *For any $\{a_i\}_{i=1}^n \subset \mathbf{C}$ satisfying $\prod_{i=1}^{n} a_i = 1$ and any $\beta > 0$ we have:*

$$\sqrt{\sum_{i=1}^{n}(\log|a_i|)^2} \leq \frac{\sqrt{n}}{\beta}\operatorname{arccosh}\left(\left(\frac{1}{n}\sum_{i=1}^{n}|a_i|^{\beta}\right)^{n-1}\right).$$

Proof. Let $\alpha = \beta\sqrt{n} > 0$.

$$\begin{aligned}
\cosh\left(\frac{\alpha}{n}\sqrt{\sum_{i=1}^{n}(\log|a_i|)^2}\right) &= \sum_{m=0}^{\infty}\frac{1}{(2m)!}\left(\sum_i\left(\frac{\alpha}{n}\log|a_i|\right)^2\right)^m \quad (18)\\
&= \sum_{m=0}^{\infty}\frac{n^{-m}}{(2m)!}\left(\frac{1}{n}\sum_i(\alpha\log|a_i|)^2\right)^m
\end{aligned}$$

$$\leq 1 + \sum_{m=1}^{\infty} \frac{n^{-m}}{(2m)!} \left(\frac{1}{n} \sum_i (\alpha \log |a_i|)^{2m} \right) \tag{19}$$

$$= \frac{1}{n} \sum_i \left(1 + \sum_{m=1}^{\infty} \frac{1}{(2m)!} \left(\frac{\alpha \log |a_i|}{\sqrt{n}} \right)^{2m} \right)$$

$$= \frac{1}{n} \sum_i \cosh \left(\log |a_i|^{\alpha \sqrt{n^{-1}}} \right)$$

$$= \frac{1}{2n} \sum_i \left(|a_i|^{\beta} + |a_i^{-1}|^{\beta} \right) \tag{20}$$

In equation (18) we used the Taylor Series expansion of $\cosh(x)$, while in (19), we used that, for $m \geq 2$ and $x \geq 0$, $f(x) = x^m$ is convex to apply Jensen's inequality: $f\left(\frac{\sum x_i}{n}\right) \leq \frac{\sum f(x_i)}{n}$.

Next we want to bound $\frac{1}{2n} \sum_i \left(|a_i|^{\beta} + |a_i^{-1}|^{\beta} \right)$. Note that for any i, $a_i^{-1} = \prod_{j \neq i} a_j$. So $\sum_i a_i^{-1} = \sum_i \prod_{j \neq i} a_j$, which is a symmetric polynomial that arises as the coefficient of the linear term in the characteristic polynomial $\prod(x - a_j)$. Let $b > 0$, then, similarly, $\sum_i |a_i|^{-b} = \sum_i \prod_{j \neq i} |a_j|^b$. We can use this and Maclaurin's Inequality [1] to bound $\sum_i |a_i|^{-b}$ above by $\sum_i |a_i|^b$. In particular,

$$\frac{1}{\binom{n}{1}} \sum_{i=1}^{n} |a_i|^b \geq \left(\frac{1}{\binom{n}{n-1}} \sum_{i=1}^{n} \prod_{j \neq i} |a_j|^b \right)^{\frac{1}{n-1}} \geq \left(\prod_{i=1}^{n} |a_i|^b \right)^{\frac{1}{n}} = 1.$$

Simplifying this,

$$\frac{1}{n} \sum_i |a_i|^b \geq \left(\frac{1}{n} \sum_i |a_i|^{-b} \right)^{\frac{1}{n-1}} \geq 1$$

so

$$\frac{1}{n^{n-2}} \left(\sum_i |a_i|^b \right)^{n-1} \geq \sum_i |a_i^{-1}|^b \geq n.$$

Using this with $b = \beta$, we can bound inequality (20) as follows:

$$\frac{1}{2n} \sum_i \left(|a_i|^{\beta} + |a_i^{-1}|^{\beta} \right) \leq \frac{1}{2n} \left[\sum_i |a_i|^{\beta} + \frac{1}{n^{n-2}} \left(\sum_i |a_i|^{\beta} \right)^{n-1} \right]$$

$$\leq \frac{1}{2n} \left[\frac{2}{n^{n-2}} \left(\sum_i |a_i|^{\beta} \right)^{n-1} \right]$$

$$= \left(\frac{1}{n} \sum_i |a_i|^{\beta} \right)^{n-1}$$

where in the last inequality we used that $\frac{1}{n} \sum_i |a_i|^{\beta} \geq 1$ and so $n(\frac{1}{n} \sum_i |a_i|^{\beta})^{n-1} \geq \sum_i |a_i|^{\beta}$. Hence, using that $\frac{\alpha}{n} = \frac{\beta}{\sqrt{n}}$, (18) is bounded above as follows:

$$\cosh \left(\frac{\beta}{\sqrt{n}} \sqrt{\sum_i (\log |a_i|)^2} \right) \leq \left(\frac{1}{n} \sum_i |a_i|^{\beta} \right)^{n-1},$$

which we can re-write to get the desired result:

$$\sqrt{\sum_i (\log|a_i|)^2} \leq \frac{\sqrt{n}}{\beta} \operatorname{arccosh}\left(\left(\frac{1}{n}\sum_i |a_i|^\beta\right)^{n-1}\right)$$

since $\operatorname{arccosh}(x)$ is increasing on $[1,\infty)$. □

For $x \in \mathrm{SL}_n(\mathbf{R})$, let

$$p_x(X) = X^n - s_1(x)X^{n-1} + s_2(x)X^{n-2} - \cdots + (-1)^{n-1}s_{n-1}(x)X + (-1)^n \tag{21}$$

be the characteristic polynomial of x where $s_j(x)$ denotes the j^{th} elementary symmetric polynomial in the eigenvalues of x (e.g. $\operatorname{tr}(x) = s_1(x)$ and $\det(x) = s_n(x) = 1$). Newton's identities [13] state that for all $1 \leq j \leq n$,

$$js_j(x) = s_{j-1}(x)\operatorname{tr}(x) - s_{j-2}(x)\operatorname{tr}(x^2) + \cdots + (-1)^{j-2}s_1(x)\operatorname{tr}(x^{j-1}) + (-1)^{j-1}\operatorname{tr}(x^j) \tag{22}$$

In particular, these recursively show that each $s_j(x)$ can be written as a linear combination of $\{\operatorname{tr}(x), \operatorname{tr}(x^2), \ldots \operatorname{tr}(x^j)\}$. Fujiwara's bound [18], applied to our context states that if λ is a root of the characteristic polynomial (21), then

$$|\lambda| \leq 2\max\left\{|s_1(x)|, |s_2(x)|^{\frac{1}{2}}, \ldots, |s_{n-1}(x)|^{\frac{1}{n-1}}, 2^{-\frac{1}{n}}\right\}. \tag{23}$$

Relationships (22) and (23) are used in the proof of the following proposition.

Proposition 3.6. *For each n, there exists a continuous function $F_n : \mathbf{R}^n \to \mathbf{R}$ such that*

$$\operatorname{tr}(x_h) \leq F_n(\operatorname{tr}(x), \operatorname{tr}(x^2), \ldots, \operatorname{tr}(x^n)).$$

One such continuous function is: $F_n(z_1, \ldots, z_n) = 2n + 2n\sum_{j=1}^n |z_j|$.

Proof. Claim: For $1 \leq k \leq n-1$, $k|s_k(x)| \leq \left(\sum_{l=1}^k |\operatorname{tr}(x^l)|\right)^k$.
This is true for $k = 1$ since $|s_1(x)| = |\operatorname{tr}(x)|$. Suppose it is true for $1 \leq k \leq j-1 < n-1$. Then, using Newton's Identities, for $k = j$:

$$\begin{aligned}
j|s_j(x)| &\leq |s_{j-1}(x)\operatorname{tr}(x)| + |s_{j-2}(x)\operatorname{tr}(x^2)| + \cdots + |s_1(x)\operatorname{tr}(x^{j-1})| + |\operatorname{tr}(x^j)| \\
&\leq \frac{|\operatorname{tr}(x)|}{j-1}\left(\sum_{l=1}^{j-1}|\operatorname{tr}(x^l)|\right)^{j-1} + \frac{|\operatorname{tr}(x^2)|}{j-2}\left(\sum_{l=1}^{j-2}|\operatorname{tr}(x^l)|\right)^{j-2} + \cdots + |\operatorname{tr}(x^{j-1})\operatorname{tr}(x)| + |\operatorname{tr}(x^j)| \\
&\leq \left(\sum_{l=1}^{j-1}|\operatorname{tr}(x^l)|\right)^{j-1}\left(\frac{|\operatorname{tr}(x)|}{j-1} + \frac{|\operatorname{tr}(x^2)|}{j-2} + \cdots + |\operatorname{tr}(x^{j-1})|\right) + |\operatorname{tr}(x^j)| \\
&\leq \left(\sum_{l=1}^{j-1}|\operatorname{tr}(x^l)|\right)^{j} + |\operatorname{tr}(x^j)| \leq \left(\sum_{l=1}^{j}|\operatorname{tr}(x^l)|\right)^{j},
\end{aligned}$$

which proves the claim. In the above inequalities we use the assumption $|\operatorname{tr}(x)| > n$, which implies $\left(\sum_{l=1}^m |\operatorname{tr}(x^l)|\right) > n$ for any $1 \leq m \leq n$. Combining this bound with (23) we get:

$$\operatorname{tr}(x_h) \leq n|\lambda| \leq 2n\max\left\{\left(\sum_{l=1}^n |\operatorname{tr}(x^l)|\right), 2^{-\frac{1}{n}}\right\}$$

We can simplify the previous bound to: $\operatorname{tr}(x_h) \leq 2n + 2n\sum_{l=1}^n |\operatorname{tr}(x^l)|$. □

Remark 3.7. Proposition 3.6 still holds with the same function F_n if we change the assumption on traces from $|\mathrm{tr}(x)| > n$ to $\mathrm{tr}(x^j) \in \mathbf{Z}$ for $1 \leq j \leq n$. Or, more generally, if we instead assume that for each $1 \leq j \leq n$, either $|\mathrm{tr}(x^j)| \geq 1$ or $|\mathrm{tr}(x^j)| = 0$.

4. Central Simple Algebras Over $\mathbf{Q}$ and Their Associated Orbifolds

Let A be a central simple algebra over $\mathbf{Q}$ of dimension $n^2 \geq 4$. By Wedderburn's structure theorem there exists a positive integer m and central division algebra D over $\mathbf{Q}$ such that $A \cong \mathrm{M}_m(D)$. Therefore $n^2 = m^2 \dim_{\mathbf{Q}}(D)$.

Suppose now that p is prime and consider the central simple algebra $A \otimes_{\mathbf{Q}} \mathbf{Q}_p \cong \mathrm{M}_m(D \otimes_{\mathbf{Q}} \mathbf{Q}_p)$ over $\mathbf{Q}_p$. This algebra also has dimension n^2 and, by Wedderburn's theorem, is isomorphic to $\mathrm{M}_{m_p}(D_p)$ for some positive integer m_p and central division algebra D_p over $\mathbf{Q}_p$. If the dimension of D_p is greater than 1 (equivalently, $m_p < n$) then we say that p *ramifies* in A. Otherwise p is *unramified* in A.

Let K be an extension field of $\mathbf{Q}$ for which there is an isomorphism of K-algebras

$$h : A \otimes_{\mathbf{Q}} K \to \mathrm{M}_n(K).$$

Given an element $x \in A \otimes_{\mathbf{Q}} K$ the characteristic polynomial of $h(x)$ is well-defined and does not depend on the isomorphism h. For an element $a \in A$, the reduced characteristic polynomial of a is defined as the characteristic polynomial of $h(a \otimes 1)$ and is of the form

$$X^n - \mathrm{tr}(a) X^{n-1} + \cdots + (-1)^n \, \mathrm{nr}(a).$$

We call $\mathrm{tr}(a)$ the *reduced trace* of a and $\mathrm{nr}(a)$ the *reduced norm* of a.

We now define orders in central simple algebras over $\mathbf{Q}$. Let A be a finite dimensional central simple algebra over $\mathbf{Q}$. A $\mathbf{Z}$*-order* $\mathcal{O}$ of A is a subring of A which is also a finitely generated $\mathbf{Z}$-submodule of A for which $\mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Q} \cong A$. An order of A is *maximal* if it is not properly contained in any other order of A. A fundamental result [27, Theorem 8.6] is that if $\mathcal{O}$ is an order of A then the reduced characteristic polynomial of an element of $\mathcal{O}$ lies in $\mathbf{Z}[X]$. In particular if $x \in \mathcal{O}$ then both the reduced trace $\mathrm{tr}(x)$ and the reduced norm $\mathrm{nr}(x)$ of x are integers.

We now discuss the construction of locally symmetric orbifolds from maximal orders in central simple algebras. Let A be a central simple algebra of dimension n^2 over $\mathbf{Q}$ for which $A \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathrm{M}_n(\mathbf{R})$, and $\mathcal{O}$ be a maximal order of A. Denote by $\mathcal{O}^1$ the multiplicative subgroup of $\mathcal{O}^\times$ consisting of those elements with reduced norm one and by Γ the image of $\mathcal{O}^1$ in $\mathrm{SL}_n(\mathbf{R})$. Defined in this manner, Γ is a lattice in $\mathrm{SL}_n(\mathbf{R})$ with finite covolume [5] (see also [32] and the references therein). Let $M_\Gamma = \Gamma \backslash \mathrm{SL}_n(\mathbf{R}) / \mathrm{SO}(n)$ be the associated special linear orbifold. This orbifold is a manifold if and only if Γ is torsion-free and is compact if and only if A is a division algebra. We call any orbifold commensurable with M_Γ *a special linear orbifold of degree n arising from a central simple algebra*.

Remark 4.1. Not every lattice in $\mathrm{SL}_n(\mathbf{R})$ arises from the aforementioned construction. In particular there exist lattices in $\mathrm{SL}_n(\mathbf{R})$ that are not commensurable with the ones coming from central simple algebras (see [30]). Nevertheless we are able to restrict our attention to the lattices arising from central simple algebras because they are universal in the sense that all other lattices virtually embed into them in a controlled way. This will be described in Section 7.

5. Trace Estimates in Congruence Subgroups

Let A be a central simple algebra over $\mathbf{Q}$ of dimension $n^2 \geq 4$, $\mathcal{O}$ be a maximal order of A and Γ be the lattice in $\mathrm{SL}_n(\mathbf{R})$ associated to the elements of $\mathcal{O}^1$. Given a natural number $N \geq 1$ we have an ideal $N\mathcal{O}$ of $\mathcal{O}$ whose quotient $\mathcal{O}/N\mathcal{O}$ is a finite ring. We define the *level N principal congruence subgroup* of $\mathcal{O}^1$ to be the kernel of the homomorphism $\mathcal{O}^1 \to (\mathcal{O}/N\mathcal{O})^\times$ obtained from the natural projection $\mathcal{O} \to \mathcal{O}/N\mathcal{O}$. We will denote this group by $\mathcal{O}^1(N)$; that is, $\mathcal{O}^1(N) = \ker\left(\mathcal{O}^1 \to (\mathcal{O}/N\mathcal{O})^\times\right)$. The image in $\mathrm{SL}_n(\mathbf{R})$ of $\mathcal{O}^1(N)$ will be denoted $\Gamma(N)$.

We now apply Theorem 3.1 to examine the growth of traces of elements of Γ in congruence subgroups.

Theorem 5.1. *Let $\Gamma \subset \mathrm{SL}_n(\mathbf{R})$ be the lattice defined above and let p be a prime which does not ramify in A and satisfies $p > 2n$. For every $m \geq 1$ and $x \in \Gamma(p^m) \setminus \{\pm 1\}$, there is an integer q, $|q| \leq \frac{n}{2}$, such that $|\mathrm{tr}(x^q)| > p^m - n$.*

Proof. Choose a basis of $A \otimes_{\mathbf{Q}} \mathbf{Q}_p$ so that $A \otimes_{\mathbf{Q}} \mathbf{Q}_p \cong \mathrm{M}_n(\mathbf{Q}_p)$ and $\mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Z}_p \cong \mathrm{M}_n(\mathbf{Z}_p)$. Denote by φ_p the natural projection $\varphi_p : \mathrm{M}_n(\mathbf{Z}_p) \to \mathrm{M}_n(\mathbf{Z}_p/p^m\mathbf{Z}_p) \cong \mathrm{M}_n(\mathbf{Z}/p^m\mathbf{Z})$. Suppose that $x \in \Gamma(p^m)$ and $x \notin \{\pm 1\}$. Identifying x with its image in $\mathrm{M}_n(\mathbf{Z}_p)$, we have that $\varphi_p(x) = \mathrm{Id}_n$, hence $\mathrm{tr}(\varphi_p(x)) = n$. Because $\mathcal{O}$ is a $\mathbf{Z}$-order of A and $x \in \mathcal{O}$ we have that $\mathrm{tr}(x) \in \mathbf{Z}$, an observation which allows us to conclude that $\mathrm{tr}(x) \equiv n \pmod{p^m}$. This shows that if $x \in \Gamma(p^m) \setminus \{\pm 1\}$ then $\mathrm{tr}(x) = p^m k + n$ for some $k \in \mathbf{Z}$.

Fix $x \in \Gamma(p^m)$ and suppose that for all integers q, $|q| \leq \frac{n}{2}$, $\mathrm{tr}(x^q) = n$. Let $p_x(X)$ be the characteristic polynomial of x as in (21). Then by Newton's identities (22), and the fact that for $x \in \mathrm{SL}_n(\mathbf{R})$ $s_i(x) = s_{n-i}(x^{-1})$, our assumptions on the traces of powers of x uniquely determines each $s_j(x)$, and a computation shows that $p_x(X) = (X-1)^n$. Since x is diagonalizable, we deduce that $x = 1$. $\square$

Corollary 5.2. *Let Γ be as above. For each $x \in \Gamma(p^m)$, $x \neq 1$,*

$$\frac{2\sqrt{2}}{n} \operatorname{arccosh}\left(\frac{p^m - n}{n}\right) \leq \ell(x). \tag{24}$$

Proof. Observe that $\ell(x) = \ell(x^{-1})$, $s_i(x) = s_{n-i}(x^{-1})$, and $\ell(x^q) = q\ell(x)$. This together with the Trace-Length Bounds Theorem 3.1 gives the result. $\square$

Remark 5.3. As only finitely many primes ramify in a central simple algebra, Theorem 5.1 hold for all but finitely many primes p.

6. Proof of Theorem A

We now prove Theorem A. Let A be a central simple algebra over $\mathbf{Q}$ of dimension $n^2 \geq 4$, $\mathcal{O}$ be a maximal order of A and Γ be the image in $\mathrm{SL}_n(\mathbf{R})$ of the multiplicative group of elements of $\mathcal{O}$ of reduced norm one. Let $M = \Gamma \backslash \mathrm{SL}_n(\mathbf{R}) / \mathrm{SO}(n)$. Given a prime p and positive integer m we denote by M_{p^m} the congruence cover of M of level p^m, and by $|M_{p^m} : M|$ the degree of M_{p^m} over M.

An immediate application of Corollary 5.2 is that

$$\mathrm{sys}(M_{p^m}) \geq \frac{2\sqrt{2}}{n} \operatorname{arccosh}\left(\frac{p^m - n}{n}\right),$$

which implies that

$$\operatorname{sys}(M_{p^m}) \geq \frac{2\sqrt{2}}{n} \log\left(\frac{p^m - n}{n}\right) \tag{25}$$

because $\operatorname{arccosh} z = \log(z + \sqrt{z^2 - 1})$.

Let S be the set of rational primes which either ramify in A or else satisfy $p < 2n$. Observe that S is a finite set. By construction, for each $p \notin S$, we have

$$|M_{p^m} : M| = |\Gamma : \Gamma(p^m)| \leq |\mathrm{SL}_n(\mathbf{Z}/p^m\mathbf{Z})| \leq (p^m)^{n^2-1}.$$

Substituting this into (25) and simplifying yields, for all $p \notin S$,

$$\begin{aligned}
\operatorname{sys}(M_{p^m}) &\geq \frac{2\sqrt{2}}{n}\left(\log\left(|M_{p^m} : M|^{1/(n^2-1)}\right) - \log(2n)\right) \\
&= \frac{2\sqrt{2}}{n}\left(\frac{1}{n^2-1}\log(\operatorname{Vol}(M_{p^m})/\operatorname{Vol}(M)) - \log(2n)\right) \\
&= \frac{2\sqrt{2}}{n(n^2-1)}\log(\operatorname{Vol}(M_{p^m})) - c,
\end{aligned}$$

where c is a positive constant depending on M. This proves Theorem A in the case that our special linear manifold M is of the form $\Gamma\backslash\mathrm{SL}_n(\mathbf{R})/\mathrm{SO}(n)$ with Γ arising from the units of norm one in a maximal order of a central simple algebra of dimension n^2 over $\mathbf{Q}$. By Proposition 2.2, the general case follows from this special case however, since by definition every special linear manifold arising from a central simple algebra is commensurable with one of the manifolds considered above.

7. Simple Locally Symmetric Orbifolds: Immersions and Towers

In the remainder of the paper, we assume familiarity with algebraic and arithmetic groups. For detailed references on these topics, we refer the reader to [4, 32]. Let k denote a totally real number field, $\mathcal{O}_k$ its ring of integers, and let G be a connected, simple, semisimple, adjoint algebraic k-group that is anisotropic at all but one real place of k. Fix once and for all the infinite embedding $k \subset \mathbf{R}$ for which G is isotropic. Then $\mathrm{G}(\mathbf{R})$ is a simple Lie group (in the sense that the complexification of its Lie algebra is simple) and $\mathrm{G}(\mathcal{O}_k) \subset \mathrm{G}(\mathbf{R})$ is a *principal arithmetic* lattice. Let $\mathrm{K} \subset \mathrm{G}(\mathbf{R})$ be a maximal compact subgroup. Then $\mathrm{G}(\mathcal{O}_k)\backslash\mathrm{G}(\mathbf{R})/\mathrm{K}$ is a *principal arithmetic simple locally symmetric orbifold.*

A *simple locally symmetric orbifold* is a Riemannian orbifold of the form $\Gamma\backslash\mathcal{G}/\mathcal{K}$ where $\mathcal{G}$ is a connected simple Lie group, $\mathcal{K}$ is a maximal compact, and Γ is a lattice. Examples of simple locally symmetric orbifolds include finite volume real, complex, and quaternionic hyperbolic manifolds, as well as special linear manifolds. Such orbifolds are *arithmetic* when M is commensurable with a principle arithmetic simple locally symmetric orbifold. By the work of Margulis [19] and Gromov–Schoen [11], all simple locally symmetric orbifolds other than real and complex hyperbolic are arithmetic.

Theorem 7.1. *Let $N = \Lambda\backslash\mathrm{G}(\mathbf{R})/\mathrm{K}$ be an arithmetic simple locally symmetric orbifold and k be its field of definition. Then there exists a space commensurable to N that can be immersed as a totally geodesic suborbifold of a special linear orbifold of degree* $[k : \mathbf{Q}]\dim\mathrm{G}$ *arising from a central simple algebra.*

Theorem 7.1 is a consequence of the following algebraic result.

Proposition 7.2. *Let* $d_1 = \dim \mathrm{G}$ *and* $d_2 = [k : \mathbf{Q}]$. *Then there exists a Lie group embedding* $\rho : \mathrm{G}(\mathbf{R}) \to \mathrm{SL}_{d_1 d_2}(\mathbf{R})$ *such that* $\rho^{-1}(\rho(\mathrm{G}(\mathbf{R})) \cap \mathrm{SL}_{d_1 d_2}(\mathbf{Z}))$ *is commensurable with* Λ.

Proof. Let $\mathfrak{g}$ denote the Lie algebra of G. The adjoint representation $\mathrm{Ad} : \mathrm{G} \to \mathrm{SL}(\mathfrak{g}) \cong \mathrm{SL}_{d_1}$ is k-rational [4, 3.13] and since G is adjoint, it is injective. Via restriction of scalars and the regular representation [24, 2.1.2], we get a sequence of $\mathbf{Q}$-rational injections:

$$\mathrm{R}_{k/\mathbf{Q}}(\mathrm{G}) \to \mathrm{R}_{k/\mathbf{Q}}(\mathrm{SL}_{d_1}) \to \mathrm{SL}_{d_1 d_2}$$

Since $\mathrm{G}(\mathbf{R})$ is naturally identified as the unique noncompact factor of $\mathrm{R}_{k/\mathbf{Q}}(\mathrm{G})(\mathbf{R})$, we obtain a sequence of Lie group injections:

$$\mathrm{G}(\mathbf{R}) \to (\mathrm{R}_{k/\mathbf{Q}}(\mathrm{G}))(\mathbf{R}) \to \mathrm{SL}_{d_1 d_2}(\mathbf{R})$$

whose composition we denote ρ. Choosing an $\mathcal{O}_k$-lattice $L \subset \mathfrak{g}$ determines a $\mathbf{Z}$-structure on $\mathrm{SL}_{d_1 d_2}$ for which $\rho^{-1}(\rho(\mathrm{G}(\mathbf{R})) \cap \mathrm{SL}_{d_1 d_2}(\mathbf{Z}))$ is commensurable with Λ [5, Prop. 6.2]. □

We record $d_1 = \dim \mathrm{G}$ as a function of absolute rank r for each Killing–Cartan type below.

	A_r	B_r	C_r	D_r	E_6	E_7	E_8	F_4	G_2
d_1	$r^2 + 2r$	$2r^2 + r$	$2r^2 + r$	$2r^2 - r$	78	133	248	52	14

For each prime p, the p-congruence tower $\{M_{p^m}\}$ of $M = \mathrm{SL}_{d_1 d_2}(\mathbf{Z}) \backslash \mathrm{SL}_{d_1 d_2}(\mathbf{R}) / \mathrm{SO}(d_1 d_2)$ induces *p-congruence tower* $\{N_{p^m}\}$ of N.

Remark 7.3. In Proposition 7.2, the $\mathbf{Q}$-structure on $\mathrm{SL}_{d_1 d_2}$ is canonical, but the $\mathbf{Z}$-structure requires a choice of an $\mathcal{O}_k$-lattice. A different choice of L would result in a commensurable cover [5, Cor. 6.3]. The two resulting induced towers over N towers are commensurable of bounded distance, and hence by Proposition 2.2 the growth up an associated p-congruence tower is independent of the choice of $\mathcal{O}_k$-lattice. Furthermore, since $(\mathrm{R}_{k/\mathbf{Q}}(\mathrm{G}))(\mathbf{Z}) \cong \mathrm{G}(\mathcal{O}_k)$ [24, 2.1.2], a p-congruence tower is of bounded distance from the tower associated to the principal congruence subgroups $\ker(\mathrm{G}(\mathcal{O}_k) \to \mathrm{G}(\mathcal{O}_k / p^m \mathcal{O}_k))$. In this sense, up to bounded distance, $\{N_{p^m}\}$ is a natural tower only depending on G and p.

Endowing N and its covers with the *subspace metric*, g, enables us to use Theorem A to obtain coarse estimates for the growth of the systoles up the $\{N_{p^m}\}$. Observe that by the definition of the canonical embedding in Proposition 7.2, the subspace metric is $2d_2$ times the Killing metric on N.

Corollary 7.4. *Let N be an arithmetic simple locally symmetric orbifold, k its field of definition and* G *its associated group. Let $d_1 = \dim \mathrm{G}$ and $d_2 = [k : \mathbf{Q}]$. There exists an explicit constant $c_1 := c_1(d_1, d_2)$ and constant $c_2 := c_2(N, g)$ such that for all primes $p > 2d_1 d_2$ and all positive integers m,*

$$\mathrm{sys}(N_{p^m}, g) \geq \frac{2\sqrt{2}}{d_1 d_2 (d_1^2 d_2^2 - 1)} \log(\mathrm{Vol}(N_{p^m}, g)) - c_2. \tag{26}$$

In light if Proposition 2.2, for every N, and sufficiently large p, systole growth is at least logarithmic in volume up every congruence p-tower, and this fact is independent of the choice of metric.

8. Arithmetic Hyperbolic Orbifolds

A special class of simple locally symmetric spaces are hyperbolic n-orbifolds. They arise from lattices in $\mathrm{SO}_0(n,1)$, as real forms of type B_r when n is even and D_r when n is odd. Relative to the canonical embedding, $\mathfrak{so}(n,1)$ is a Lie subalgebra of $\mathfrak{sl}_{n+1}(\mathbf{R})$ and the tangent space of hyperbolic n-space $\mathrm{SO}_0(n,1)/\mathrm{SO}(n)$ can be identified with

$$\left\{ \left(\begin{array}{cccc:c} & & & & x_1 \\ & & 0 & & x_2 \\ & & & & \vdots \\ & & & & x_n \\ \hdashline x_1 & x_2 & \dots & x_n & 0 \end{array} \right) \;\middle|\; x_1, \dots, x_n \in \mathbf{R} \right\}$$

The Killing form on $\mathfrak{so}(n,1)$ is $(n-1)\operatorname{tr}(XY)$ and a direct computation [12, V.3.1] shows that the Killing metric has sectional curvature $-\frac{1}{2(n-1)}$. Specializing the results of the previous section yields the following.

Corollary 8.1. *Let N be an arithmetic hyperbolic n-orbifold with field of definition k of degree d. Then N is commensurable to an immersed totally geodesic subspace of a degree $d(2n^2+5n+3)$ special linear orbifold. With respect to the subspace metric, this subspace has constant sectional curvature $-\frac{1}{4(n-1)d}$*

Before stating the following corollary, we recall that hyperbolic orbifolds will be given the hyperbolic metric in which they have constant sectional curvature -1 unless an alternative Riemannian metric g is explicitly given.

Corollary 8.2. *Let N be an arithmetic hyperbolic n-orbifold, $n \geq 4$, with field of definition k of degree d. There exists a constant $c_2 := c_2(N)$ such that for all primes $p > 2(2n^2+5n+3)d$ and all positive integers m,*

$$\operatorname{sys}(N_{p^m}) \geq \frac{\sqrt{2}}{144 d^{7/2} n^{7/2}} \log(\operatorname{Vol}(N_{p^m})) - c_2. \tag{27}$$

Proof. Let g denote the subspace metric on N and its covers. By scaling conversions (9) and Theorem A,

$$\begin{aligned} \operatorname{sys}(N_{p^m}) &= \sqrt{\frac{1}{4(n-1)d}} \operatorname{sys}(N_{p^m}, g) \\ &\geq \sqrt{\frac{1}{4(n-1)d}} \left(\frac{2\sqrt{2}}{((2n^2+5n+3)d)((2n^2+5n+3)d)^2-1)} \right) \log(\operatorname{Vol}(N_{p^m}), g) - c_2' \\ &\geq \left(\frac{\sqrt{2}}{d^{7/2} n (2n^2+5n+3)^3} \right) \log(\operatorname{Vol}(N_{p^m}), g) - c_2' \\ &\geq \frac{\sqrt{2}}{144 d^{7/2} n^{7/2}} \log(\operatorname{Vol}(N_{p^m})) - c_2. \end{aligned}$$

$\square$

Standard arithmetic hyperbolic orbifolds are particularly simple and enable a strengthening of the above results.

Proof of Theorem B. Begin by supposing that N is principal arithmetic. Then $\mathrm{G} = \mathrm{PSO}(V,q)$ where (V,q) is an $(n+1)$-dimensional quadratic space over $\mathbf{Q}$ with signature $(n,1)$ over $\mathbf{R}$ and the canonical embedding $\mathrm{SO}(V,q) \to \mathrm{SL}(V) \cong \mathrm{SL}_{n+1}$ is $\mathbf{Q}$-rational. It follows that $N_{p^m} = \Lambda(p^m)\backslash \mathrm{SO}_0(n,1)/\mathrm{SO}(n)$ where $\Lambda(p^m) := \Gamma(p^m) \cap \mathrm{SO}_0(n,1)$. Without loss of generality, we may assume $q = \langle a_1, \dots a_n, -a_{n+1}\rangle$ where each a_i is a positive integer. Let S be the set of odd primes that excludes the finitely many which divide the a_i. For each $p \in S$,

$$|N_{p^m} : N| = |\Lambda : \Lambda(p^m)| \leq |\mathrm{SO}_{n+1}(\mathbf{Z}/p^m\mathbf{Z})| \leq (p^m)^{\dim \mathrm{SO}_{n+1}}.$$

With respect to the subspace metric g (10), (N_{p^m}, g) is an isometrically immersed totally geodesics subspace of M_{p^m}, in particular, $\mathrm{sys}(N_{p^m}, g) \geq \mathrm{sys}(M_{p^m}, g)$. Using the systole bound (25):

$$\begin{aligned}
\mathrm{sys}(N_{p^m}, g) &\geq \frac{2\sqrt{2}}{n+1}\left(\log\left(|N_{p^m} : N|^{2/n(n+1)}\right) - \log(2n+2)\right) \\
&= \frac{2\sqrt{2}}{n+1}\left(\frac{2}{n(n+1)}\log(\mathrm{Vol}(N_{p^m})/\mathrm{Vol}(N)) - \log(2n+2)\right) \\
&= \frac{4\sqrt{2}}{n(n+1)^2}\log(\mathrm{Vol}(N_{p^m})) - c.
\end{aligned}$$

A direct computation shows that the sectional curvature [12, V.3.1] of (N_{p^m}, g) is $-\frac{1}{4}$, and hence the metric of constant sectional curvature -1 is $\frac{1}{4}g$. By (9), $\mathrm{sys}(N_{p^m}) = \frac{1}{\sqrt{4}}\,\mathrm{sys}(N_{p^m}, g)$, which proves the growth bounds for principle standard arithmetic hyperbolic orbifolds. The general case follows from Proposition 2.2. □

9. Arithmetic Measure and the Proofs of Theorem C and Theorem D

In the previous sections, we established that for all but finitely many primes p, the systole growth up p-congruence covers is at least logarithmic in metric volume, and furthermore, we explicitly computed the multiplicative constant c_1 in terms of the algebraic data $\dim \mathrm{G}$ and $[k : \mathbf{Q}]$. In this section, we show how this algebraic data can be replaced with the geometric data of dimension and volume of N. It is a direct computation for each $\mathrm{G}(\mathbf{R})$ to write $\dim \mathrm{G}$ as a function of $\dim N$, see for example [12, Ch. X. §6. Table V]. Furthermore, the propositions of this section bound $[k : \mathbf{Q}]$ by an explicit function volume. Theorems C and D then follow from Propositions 9.2 and 9.3, respectively. The propositions are highly technical and we assume familiarity with Prasad's volume formula [25].

Let N, G, k, and r be as in the previous section and let $\widetilde{\mathrm{G}}$ denote the simply connected cover of G. If $\iota : \widetilde{\mathrm{G}} \to \mathrm{G}$ is the central isogeny, then we may lift the lattice $\pi_1(N) \subset \mathrm{G}(\mathbf{R})$ to a lattice $\Lambda_N := \iota^{-1}(\pi_1(N)) \subset \widetilde{\mathrm{G}}(\mathbf{R})$. It follows that $\Lambda_N\backslash\widetilde{\mathrm{G}}(\mathbf{R}) \cong \pi_1(N)\backslash\mathrm{G}(\mathbf{R})$. By the *arithmetic measure* μ_a on N we mean the pushforward of Prasad's normalized Haar measure μ_∞ on $\Lambda_N\backslash\widetilde{\mathrm{G}}(\mathbf{R})$ [25, 3.11]. In particular, $\mu_a(N) := \mu_\infty(\Lambda_N\backslash\widetilde{\mathrm{G}}(\mathbf{R}))$.

Remark 9.1. As $\frac{\mathrm{Vol}(N_{p^m},g)}{\mathrm{Vol}(N,g)} = \frac{\mu_a(N_{p^m})}{\mu_a(N)}$, it follows that $\log(\mathrm{Vol}(N_{p^m}, g)) = \log(\mu_a(N_{p^m})) + C$ where $C = C(N)$. As such, Theorems A and B and Corollaries 7.4 and 8.2 hold equally well when metric volume is replaced with arithmetic measure.

Let $m_1 \leq m_2 \leq \cdots \leq m_r$ be the exponents of the simple, simply connected, compact real-analytic Lie group of the same type as G. These exponents can be found in [6]. Given these exponents, we define a function $f(m_1, \ldots, m_r)$ by

$$f(m_1, \ldots, m_r) = \prod_{i=1}^{r} \frac{m_i!}{(2\pi)^{m_i+1}}.$$

(28)

Killing-Cartan Type	Exponents	Lower bound for $f(m_1, \ldots, m_r)$
A_r	$1, 2, \ldots, r$	10^{-32}
B_r	$1, 3, 5, \ldots, 2r-1$	10^{-16}
C_r	$1, 3, 5, \ldots, 2r-1$	10^{-16}
D_r	$1, 3, 5, \ldots, 2r-3, r-1$	10^{-19}
E_6	$1, 4, 5, 7, 8, 11$	10^{-15}
E_7	$1, 5, 7, 9, 11, 13, 17$	10^{-13}
E_8	$1, 7, 11, 13, 17, 19, 23, 29$	$8434.1205\cdots$
F_4	$1, 5, 7, 11$	10^{-9}
G_2	$1, 5$	10^{-5}

Proposition 9.2. *Let N be an arithmetic simple locally symmetric space of dimension n, arithmetic measure less than v, and such that Λ_N is contained in a principal arithmetic subgroup of $\widetilde{\mathrm{G}}(\mathbf{R})$. Let k denote the field of definition of N. Then the degree of k is less than $c\log(v)$ where c is the lower bound for $f(m_1, \ldots, m_r)$ given in (28). In particular $[k:\mathbf{Q}] \leq 10^{32}\log(v)$.*

Proof. We may assume without loss of generality that Λ_N is a principal arithmetic subgroup of $\widetilde{\mathrm{G}}(\mathbf{R})$. Prasad's formula [25, Theorem 3.7] for the covolume of a principal S-arithmetic subgroup implies that

$$v \geq \mathrm{D}_k^{\frac{1}{2}\dim \mathrm{G}} \left(\mathrm{D}_\ell/\mathrm{D}_k^{[\ell:k]}\right)^{\frac{1}{2}\mathfrak{s}(\mathscr{G})} f(m_1, \ldots, m_r)^{[k:\mathbf{Q}]}\mathscr{E}, \tag{29}$$

where all notation is as in Section 3 of [25]. In particular we note that D_k is the absolute value of the discriminant of k and D_ℓ is the absolute value of the discriminant of a certain finite degree extension ℓ of k. We have omitted the Tamagawa number $\tau_k(\widetilde{\mathrm{G}})$ in the above formula (which appears in the formula of Prasad) as the work of Kottwitz [15] shows that $\tau_k(\widetilde{\mathrm{G}}) = 1$ whenever k is a number field.

We now obtain a lower bound for (29). First, observe that $\mathrm{D}_k \geq 1$ and that $\mathrm{D}_\ell/\mathrm{D}_k^{[\ell:k]}$ is the norm from k to $\mathbf{Q}$ of the relative discriminant of the extension ℓ/k. Thus $\left(\mathrm{D}_\ell/\mathrm{D}_k^{[\ell:k]}\right) \geq 1$. Finally, $\mathscr{E} \geq 1$ by [25, Remark 3.10]. It follows that

$$v \geq f(m_1, \ldots, m_r)^{[k:\mathbf{Q}]}$$

so that the proposition follows from (28). □

The following proposition shows that when N is an arithmetic hyperbolic orbifold we can obtain the same bound as in Proposition 9.2 without the assumption that Λ_N is contained in a principal arithmetic group.

Proposition 9.3. *Let N be an arithmetic hyperbolic orbifold of dimension $n \geq 4$ and hyperbolic volume less than V. Let k denote the field of definition of N. Then the degree of k is less than $c\log(V)$ where $c > 0$ is an absolute, effectively computable constant.*

Proof. We may assume without loss of generality that Λ_N is a maximal arithmetic group.

We begin by considering the case in which $n = 2r$ is even. In Section 3 of [2] Belolipetsky used Prasad's volume formula to prove that

$$V \geq \frac{(2\pi)^r}{1\cdot 3\cdots(2r-1)} \cdot \frac{4\mathrm{D}_k^{\frac{1}{2}\dim \mathrm{G}-1}}{10^2\left(\frac{\pi^2}{6}\right)^{[k:\mathbf{Q}]}} \cdot f(m_1,\ldots,m_r)^{[k:\mathbf{Q}]}. \tag{30}$$

Because G is of type B_r in this case the exponents $m_1,\ldots,m_r$ are equal to $1,3,5,\ldots,2r-1$ and $f(m_1,\ldots,m_r) \geq 10^{-16}$ by (28). As we also have $\mathrm{D}_k \geq 1$ and $\dim \mathrm{G} > 1$ we may simplify (30) so as to obtain

$$[k:\mathbf{Q}] \leq c_1 \log(V) \tag{31}$$

where c_1 is a positive constant depending on r.

We now consider the case in which $n = 2r-1$ is odd.

When r is odd Belolipetsky and Emery [3, Section 7] have shown that

$$V \geq \frac{4\pi^r}{(r-1)!} \cdot \frac{\mathrm{D}_k^{r^2-r/2-2}\left(\frac{1}{2}\left(\frac{12}{\pi}\right)^2 f(m_1,\ldots,m_r)\right)^{[k:\mathbf{Q}]}}{32}. \tag{32}$$

In this case G is of type D_r, hence (28) shows that $f(m_1,\ldots,m_r) \geq 10^{-19}$. As we also have $\mathrm{D}_k^{r^2-r/2-2} \geq 1$, we may simplify (32) so as to obtain

$$[k:\mathbf{Q}] \leq c_2 \log(V) \tag{33}$$

where c_2 is a positive constant depending on r.

The case in which r is even can be handled similarly using Section 8 of [3] and considering a variety of special cases (in this case admissible groups are of type ${}^2\mathrm{D}_{\frac{r-1}{2}}$ or ${}^{3,6}\mathrm{D}_4$). ☐

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