

BRAUER EQUIVALENT NUMBER FIELDS AND THE GEOMETRY OF QUATERNIONIC SHIMURA VARIETIES

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ABSTRACT. Two number fields are said to be Brauer equivalent if there is an isomorphism between their Brauer groups that commutes with restriction. In this paper we prove a variety of number theoretic results about Brauer equivalent number fields (e.g., they must have the same signature). These results are then applied to the geometry of certain arithmetic locally symmetric spaces. As an example, we construct incommensurable arithmetic locally symmetric spaces containing exactly the same set of proper immersed totally geodesic surfaces.

1. INTRODUCTION

Given a Riemannian manifold M it is natural to ask the extent to which the geometry of M is dependent on the geodesics or totally geodesic submanifolds of M . This problem has received a considerable amount of attention in the case that M is an arithmetic manifold. With respect to geodesics, it is a theorem of Reid [15] that two arithmetic hyperbolic surfaces with the same geodesic length spectra are necessarily commensurable. This result was later generalized by Chinburg, Hamilton, Long and Reid [3] to hyperbolic 3-manifolds, and by Prasad and Rapinchuk [14] to a very broad class of locally symmetric spaces.

More recently, McReynolds and Reid [11] have proven that if two arithmetic hyperbolic 3-manifolds have the same (nonempty) set of totally geodesic surfaces (up to commensurability) then they must in fact be commensurable. In a followup paper, McReynolds [10] showed that this behavior does not persist when one considers arithmetic manifolds with universal cover the symmetric space of products of $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{C})$. In particular McReynolds constructed examples of incommensurable arithmetic manifolds with precisely the same commensurability classes of totally geodesic submanifolds arising from a fixed field. McReynolds' proof was largely algebraic and reduced to constructing central simple algebras defined over number fields which were not isomorphic yet nevertheless contained precisely the same isomorphism classes of subalgebras defined over a fixed field.

To make all of this concrete, let k be a number field with ring of integers \mathcal{O}_k and A be a central simple algebra over k so that

$$A \otimes_{\mathbb{Q}} \mathbb{R} \cong M_n(\mathbb{R})^a \times M_n(\mathbb{C})^b \times M_{n/2}(\mathbb{H})^c$$

for nonnegative integers a, b, c . Let \mathcal{O} be an \mathcal{O}_k -order of A and \mathcal{O}^1 the multiplicative subgroup consisting of those elements of \mathcal{O} with reduced norm 1. From the above isomorphism we obtain an embedding of \mathcal{O}^1 into $G_{a,b} = \mathrm{SL}_n(\mathbb{R})^a \times \mathrm{SL}_n(\mathbb{C})^b$. Let Γ denote the image of \mathcal{O}^1 under this embedding. Then Γ is an arithmetic lattice in $G_{a,b}$ and we obtain a locally symmetric space $X_{a,b}/\Gamma$, where $X_{a,b}$ is the symmetric space associated to $G_{a,b}$. We note that the commensurability class of $X_{a,b}/\Gamma$ is given by the isomorphism class of A . In light of this construction it is clear that in order to construct arithmetic manifolds with precisely the same commensurability classes of totally geodesic submanifolds arising from a fixed field it suffices to construct central simple algebras A_1

over k_1 and A_2 over k_2 with the property that for some field $F \subseteq k_1 \cap k_2$, a central simple algebra B over F satisfies $B \otimes_F k_1 \cong A_1$ if and only if $B \otimes_F k_2 \cong A_2$. Much of McReynolds' paper was devoted to constructing algebras with this property.

In a recent paper, the author, McReynolds and Miller [7] showed that one can construct algebras A_1 and A_2 with the aforementioned properties from any pair of number fields with isomorphic adèle rings. Before describing this construction we need to establish some notation. Given a number field K we denote the Brauer group of K by $\text{Br}(K)$ and note that associated to any subfield $F \subset K$ there is a restriction homomorphism $\text{Res}_{K/F}([B]) = [B \otimes_F K]$. Suppose now that k_1 and k_2 are number fields whose adèle rings are isomorphic. In [7] it was shown that there is an isomorphism $\Phi_{\text{Br}} : \text{Br}(k_1) \rightarrow \text{Br}(k_2)$ with the property that for any field $F \subset k_1 \cap k_2$ and $L \supset k_1, k_2$ the following diagram commutes:

$$(1) \quad \begin{array}{ccc} & \text{Br}(L) & \\ \text{Res} \nearrow & & \nwarrow \text{Res} \\ \text{Br}(k_1) & \xleftrightarrow{\Phi_{\text{Br}}} & \text{Br}(k_2) \\ \text{Res} \nwarrow & & \nearrow \text{Res} \\ & \text{Br}(F) & \end{array}$$

In particular this shows that we may obtain central simple algebras over k_1 and k_2 having the same isomorphism classes of subalgebras defined over F by considering the pair $[A], \Phi_{\text{Br}}([A])$ for any $[A] \in \text{Br}(k_1)$. Call two number fields k_1 and k_2 **Brauer equivalent** if there is an isomorphism $\Phi_{\text{Br}} : \text{Br}(k_1) \rightarrow \text{Br}(k_2)$ for which the diagram in (1) commutes for all $F \subset k_1 \cap k_2$ and $L \supset k_1, k_2$.

This paper studies Brauer equivalent number fields with an eye towards geometric applications. Before describing this paper's results, we recall that the Galois kernel of a number field k is the largest subfield of k which is Galois over \mathbb{Q} . Algebraically, the main result of the paper is:

Theorem 1.1. *Let k and k' be Brauer equivalent number fields. The following invariants of k and k' coincide:*

- (i) *Their degrees.*
- (ii) *Their signatures.*
- (iii) *Their Galois kernels.*
- (iv) *Their groups of roots of unity.*

As an immediate consequence of Theorem 1.1 we obtain:

Corollary 1.2. *If k and k' are Brauer equivalent number fields and k/\mathbb{Q} is Galois then $k \cong k'$.*

These results have a number of geometric consequences for locally symmetric spaces arising from quaternion algebras. In Corollary 4.3, for instance, we prove that if M_1 and M_2 are incommensurable arithmetic hyperbolic 3-manifolds with Brauer equivalent fields of definition k_1 and k_2 then M_1 and M_2 have no proper immersed totally geodesic surfaces in common. On the other hand, in Theorem 6.1 we construct incommensurable arithmetic locally symmetric spaces containing exactly the same set of proper immersed totally geodesic surfaces. This provides a striking contrast to the behavior of arithmetic hyperbolic 3-manifolds and refines the work of McReynolds [10], where the manifolds constructed were only shown to have the same commensurability classes of totally geodesic submanifolds.

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2. PRELIMINARIES AND NOTATION

In this section we will describe the notation that will be used throughout this paper.

Let k be a number field with ring of integers \mathcal{O}_k . We will denote the number of real (respectively complex) places of k by r_1 (respectively r_2). The signature of k is the pair (r_1, r_2) . Note that if $n = [k : \mathbb{Q}]$ then $n = r_1 + 2r_2$. We denote by \mathcal{P}_k the set of all primes (finite or infinite) of k . We will denote the Galois closure of k by \widehat{k} and the Galois kernel of k by k_0 . We remind the reader that the Galois kernel of a number field k/\mathbb{Q} is the largest subfield of k which is Galois over \mathbb{Q} .

Given an extension L/k of number fields and a prime $\mathfrak{P} \in \mathcal{P}_L$ there is a unique prime $\mathfrak{p} \in \mathcal{P}_k$ lying beneath \mathfrak{P} (i.e., $\mathfrak{P} \cap \mathcal{O}_k = \mathfrak{p}$). In this situation we write $\mathfrak{P} \mid \mathfrak{p}$.

We will denote the Brauer group of k by $\text{Br}(k)$, and if L is a finite degree extension of k then we will denote the relative Brauer group of L/k by $\text{Br}(L/k)$. This is defined to be the kernel of the restriction map $\text{Res}_{L/k} : \text{Br}(k) \rightarrow \text{Br}(L)$.

We will typically use A or B to denote central simple algebras over k and D to denote a central division algebra over k . Given a central simple algebra A over k there is a unique central division algebra D over k and nonnegative integer r such that $A \cong M_r(D)$. Therefore

$$\dim_k(A) = r^2 \cdot \dim_k(D).$$

We define the **degree** of A to be $\sqrt{\dim_k(A)}$ and denote it by $[A : k]$. We define the (Schur) **index** of A , denoted $\text{ind}(A)$, to be $[D : k]$. That is, the index of A is the degree of its underlying central division algebra.

Given a central simple algebra A , we will denote by A^\times the multiplicative group of invertible elements of A and by A^1 the multiplicative group of elements of A with reduced norm 1. Let A be a central simple algebra over k of dimension n^2 . Given a prime $\mathfrak{p} \in \mathcal{P}_k$ we have an associated central simple algebra $A_{\mathfrak{p}} = A \otimes_k k_{\mathfrak{p}}$ over $k_{\mathfrak{p}}$. If $A_{\mathfrak{p}} \cong M_n(k_{\mathfrak{p}})$ then we say that the prime \mathfrak{p} is **unramified** (or split) in A . Otherwise $A_{\mathfrak{p}} \cong M_m(D_{\mathfrak{p}})$ for some positive integer $m < n$ and central division algebra $D_{\mathfrak{p}}$ over $k_{\mathfrak{p}}$. In this case we say that \mathfrak{p} is **ramified** in A . We will denote by $\text{Ram}(A)$ the set of primes of k which ramify in A , by $\text{Ram}_f(A)$ the set of finite primes of k ramifying in A and by $\text{Ram}_\infty(A)$ the set of infinite primes of k that ramify in A .

We now discuss the local invariants of A . Let $\mathfrak{p} \in \mathcal{P}_k$. We begin with the case that \mathfrak{p} is an infinite prime of k so that $k_{\mathfrak{p}} = \mathbb{R}$ or $k_{\mathfrak{p}} = \mathbb{C}$. In this case we define the local invariant of A at \mathfrak{p} , denoted $\text{inv}_{\mathfrak{p}}(A)$, by $\text{inv}_{\mathfrak{p}}(A) = 0$ when $k_{\mathfrak{p}} = \mathbb{C}$, $\text{inv}_{\mathfrak{p}}(A) = 0$ when $k_{\mathfrak{p}} = \mathbb{R}$ and $A_{\mathfrak{p}} \cong M_n(\mathbb{R})$ and $\text{inv}_{\mathfrak{p}}(A) = \frac{1}{2}$ when $k_{\mathfrak{p}} = \mathbb{R}$ and $A_{\mathfrak{p}} \cong M_{\frac{n}{2}}(\mathbb{H})$. Now suppose that \mathfrak{p} is a finite prime of k . In this case there is an isomorphism $\text{inv} : \text{Br}(k_{\mathfrak{p}}) \rightarrow \mathbb{Q}/\mathbb{Z}$ (see [16, Theorem 31.8]). We define the invariant of A at \mathfrak{p} by $\text{inv}_{\mathfrak{p}}(A) = \text{inv}([A_{\mathfrak{p}}])$. Finally, if $L_{\mathfrak{P}}/k_{\mathfrak{p}}$ is a finite extension and $A_{\mathfrak{p}}$ is a central simple algebra over $k_{\mathfrak{p}}$ then by [16, Theorem 31.9] we have

$$\text{inv}_{\mathfrak{P}}(A_{\mathfrak{p}} \otimes_{k_{\mathfrak{p}}} L_{\mathfrak{P}}) = [L_{\mathfrak{P}} : k_{\mathfrak{p}}] \cdot \text{inv}_{\mathfrak{p}}(A_{\mathfrak{p}}).$$

2.1. Quaternionic Shimura varieties. In this subsection we review the construction of arithmetic lattices acting on products of hyperbolic planes and 3-dimensional hyperbolic spaces. For a complete treatment we refer the reader to [1, Section 3].

Let k be a number field with ring of integers \mathcal{O}_k and signature (r_1, r_2) and recall that $[k : \mathbb{Q}] = r_1 + 2r_2$. If A is a quaternion algebra over k which is not totally definite (i.e., there exists an infinite prime of k which splits in A) then there exist non-negative integers r, s with $r + s = r_1$ such that

$$A \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})^s \times \mathbb{H}^r \times M_2(\mathbb{C})^{r_2}.$$

This isomorphism induces an embedding

$$\pi : A^\times \hookrightarrow \prod_{\nu \notin \text{Ram}_\infty(A)} (A \otimes_k k_\nu)^\times \longrightarrow \text{GL}_2(\mathbb{R})^s \times \text{GL}_2(\mathbb{C})^{r_2}.$$

Restricting to the subgroup A^1 of elements of A with reduced norm 1 yields an embedding

$$\pi : A^1 \hookrightarrow \text{SL}_2(\mathbb{R})^s \times \text{SL}_2(\mathbb{C})^{r_2}.$$

If \mathcal{O} is a maximal order of A and \mathcal{O}^1 the multiplicative group of those elements of \mathcal{O} with reduced norm 1 then the image $\pi(\mathcal{O}^1)$ of \mathcal{O}^1 in $\text{SL}_2(\mathbb{R})^s \times \text{SL}_2(\mathbb{C})^{r_2}$ is a lattice by the work of Borel and Harish-Chandra [2]. We will denote this lattice by $\Gamma_{\mathcal{O}}$. We define an irreducible lattice in $\text{SL}_2(\mathbb{R})^s \times \text{SL}_2(\mathbb{C})^{r_2}$ to be **arithmetic** if it is commensurable with a lattice of the form $\Gamma_{\mathcal{O}}$. Note that the arithmetic lattice will be cocompact if and only if its associated quaternion algebra is not isomorphic to the matrix algebra $M_2(k)$.

Let $G = \text{SL}_2(\mathbb{R})^s \times \text{SL}_2(\mathbb{C})^{r_2}$ and K be a maximal compact subgroup of G . If Γ is an arithmetic lattice of G then the quotient space $M = \Gamma \backslash G / K$ is an arithmetic locally symmetric space called a **quaternionic Shimura variety**. We call k the **invariant trace field** of M and A the **invariant quaternion algebra** of M .

If M_1 and M_2 are quaternionic Shimura varieties then we say that M_1 and M_2 are **commensurable** if they share an isometric finite-sheeted covering. It is well-known that M_1 and M_2 are commensurable if and only if they have isomorphic invariant trace fields and invariant quaternion algebras (c.f. [9, Theorem 8.4.1]).

There are two special cases of the above construction that are worth highlighting. If k is a totally real number field for which $G = \text{SL}_2(\mathbb{R})$ (i.e., $r = r_1 - 1$ and $s = 1$) then we call any lattice commensurable with $\Gamma_{\mathcal{O}}$ an **arithmetic Fuchsian group**. If $\Gamma_{\mathcal{O}}$ is torsion-free then the resulting quotient space M is an **arithmetic hyperbolic surface**. If k is a number field containing a unique complex place and $G = \text{SL}_2(\mathbb{C})$ (i.e., $r = r_1$ and $s = 0$) then we call any lattice commensurable with $\Gamma_{\mathcal{O}}$ an **arithmetic Kleinian group** and M an **arithmetic hyperbolic 3-manifold** when $\Gamma_{\mathcal{O}}$ is torsion-free.

3. PROOF OF THEOREM 1.1

We will begin by proving that if k and k' are Brauer equivalent then $[k : \mathbb{Q}] = [k' : \mathbb{Q}]$. Let L be a number field containing k and k' such that L/k and L/k' are both Galois extensions. Because k and k' are Brauer equivalent there exists an isomorphism $\Phi_{\text{Br}} : \text{Br}(k) \rightarrow \text{Br}(k')$ for which the following diagram commutes:

$$(2) \quad \begin{array}{ccc} & \text{Br}(L) & \\ \text{Res}_L \nearrow & & \nwarrow \text{Res}_L \\ \text{Br}(k) & \xrightarrow{\Phi_{\text{Br}}} & \text{Br}(k') \end{array}$$

In particular this implies that $\Phi_{\text{Br}}(\text{Br}(L/k)) = \text{Br}(L/k')$. Let D be a division algebra over k of degree $[D : k] = [L : k]$ such that $[D] \in \text{Br}(L/k)$. By [13, Chapter 13.5], there exists a central simple algebra S over k' with $[S : k'] = [L : k']$ and $S \in \Phi_{\text{Br}}([D])$. Observe that

$$\begin{aligned} [L : k] &= [D : k] \\ &= \text{ind}([D]) \\ &= \exp([D]) \\ &= \exp(\Phi_{\text{Br}}([D])) \\ &= \text{ind}(\Phi_{\text{Br}}([D])) \\ &\leq [S : k] \\ &= [L : k']. \end{aligned}$$

Here we have used the fact that if K is a number field and $[A] \in \text{Br}(K)$ then $\text{ind}([A]) = \exp([A])$ [16, Theorem 32.19], where $\text{ind}([D])$ is the Schur index of D and $\exp([D])$ is the order of $[D]$ in $\text{Br}(K)$. This shows that $[L : k] \leq [L : k']$. Interchanging the roles of k and k' proves that $[L : k] = [L : k']$. We now see that

$$[k : \mathbb{Q}] = [L : \mathbb{Q}]/[L : k] = [L : \mathbb{Q}]/[L : k'] = [k' : \mathbb{Q}].$$

That k and k' have the same degrees and isomorphic Brauer groups imply that their signatures are the same, since the exact sequence of Brauer groups in class field theory [16, (32.13)] implies that if $\text{Br}(k) \cong \text{Br}(k')$ then k and k' have the same number of real places.

We now show that k and k' have the same Galois kernels. Let k_0 denote the Galois kernel of k and k'_0 denote the Galois kernel of k' . Let \widehat{k} denote the Galois closure over \mathbb{Q} of k . We claim that $k'_0 \subset \widehat{k}$. To see this we will use a fact proven in [7, Theorem 1.4]: If k and k' are Brauer equivalent number fields then for every prime p which is unramified in both k and k' we have

$$\gcd(f(v_1/p), \dots, f(v_g/p)) = \gcd(f(v'_1/p), \dots, f(v'_{g'}/p)),$$

where v_1, \dots, v_g are the places of K lying above p and $v'_1, \dots, v'_{g'}$ are the places of k' lying above p . Here $f(v_i/p)$ (respectively $f(v'_i/p)$) is the inertia degree of v_i (respectively v'_i) over p . Suppose that p is a prime that is unramified in k and k' and splits completely in \widehat{k}/\mathbb{Q} . Then p also splits completely in k/\mathbb{Q} , hence $\gcd(f(v_1/p), \dots, f(v_g/p)) = \gcd(f(v'_1/p), \dots, f(v'_{g'}/p)) = 1$. Let v'_0 be a prime of k'_0 lying above p . Then $f(v'_0/p)$ divides $f(v'/p)$ for all primes v' of k' lying above p , hence $f(v'_0/p) = 1$. Because k'_0/\mathbb{Q} is Galois this implies that p splits completely in k'_0/\mathbb{Q} . It follows from [5, Theorem 9, p. 168] that k'_0 is contained in \widehat{k} , proving the claim.

Brauer's theorem [12, p. 135] now implies that if every prime p having a degree one factor in k splits completely in k'_0/\mathbb{Q} (up to a set of primes with density 0) then $k'_0 \subset k$. In our situation having a degree one factor in k forces

$$\gcd(f(v_1/p), \dots, f(v_g/p)) = 1 = \gcd(f(v'_1/p), \dots, f(v'_{g'}/p)),$$

and consequently implies that p must split completely in k'_0/\mathbb{Q} . This shows that $k'_0 \subset k$. Since k_0 and k'_0 are both Galois extensions of \mathbb{Q} contained in k , their compositum must be a Galois extension of \mathbb{Q} contained in k . Since k_0 is by definition the largest Galois extension of \mathbb{Q} contained in k , this implies that the compositum of k_0 and k'_0 is equal to k_0 , hence $k'_0 \subset k_0$. By symmetry $k_0 \subset k'_0$ and $k_0 = k'_0$.

That k and k' have the same groups of roots of unity follows from their having the same Galois kernels because the roots of unity contained in each field generates a Galois extension.

4. TOTALLY GEODESIC SURFACES OF ARITHMETIC HYPERBOLIC 3-MANIFOLDS

In this section we prove that incommensurable arithmetic hyperbolic 3-manifolds defined over Brauer equivalent number fields never have any proper immersed totally geodesic surfaces in common. Our proof will make use of the following algebraic result.

Theorem 4.1. *Suppose that F is a number field and B_0/F is a quaternion algebra. Let $k, k'/F$ be Galois extensions of degree 2^n and define $A_1 = B_0 \otimes_F k$ and $A_2 = B_0 \otimes_F k'$. If $B \otimes_F k \cong A_1$ if and only if $B \otimes_F k' \cong A_2$ for every quaternion algebra B/F with $\text{Ram}_\infty(B) = \text{Ram}_\infty(B_0)$ then $k \cong k'$ and $A_1 \cong A_2$.*

Proof. Let ν_1, ν_2 be primes of F which do not split completely in k/F and do not ramify in B_0 . We will show that ν_1 and ν_2 do not split completely in k'/F either. It follows that the set of primes of F which split completely in k'/F is (with at most a finite number of exceptions) a subset of the set of primes of F which split completely in k/F . A standard consequence of the Chebotarev density theorem (cf. [5, Thm 9, p. 168]) then shows that k is isomorphic to a subfield of k' , hence the two fields are themselves isomorphic since their degrees over F are equal.

Define a quaternion algebra B over F by setting $\text{Ram}_\infty(B) = \text{Ram}_\infty(B_0)$ and $\text{Ram}_f(B) = \text{Ram}_f(B_0) \cup \{\nu_1, \nu_2\}$. We claim that if ω is a prime of k' lying above ν_1 then $\text{inv}_\omega(A_2) = 0$ in \mathbb{Q}/\mathbb{Z} . Indeed, this follows from

$$(3) \quad \begin{aligned} \text{inv}_\omega(A_2) &= \text{inv}_\omega(B_0 \otimes_F k') \\ &= \text{inv}_{\nu_1}(B_0) \cdot [k'_\omega : F_{\nu_1}] \\ &= [k'_\omega : F_{\nu_1}], \end{aligned}$$

where the last equality follows from the fact that $\nu_1 \notin \text{Ram}(B_0)$, hence $\text{inv}_{\nu_1}(B_0)$ is trivial in \mathbb{Q}/\mathbb{Z} . Since $[k'_\omega : F_{\nu_1}]$ is an integer, the claim follows.

We now show that $B \otimes_F k \cong A_1$. It suffices to show that $\text{Ram}_f(B \otimes_F k) = \text{Ram}_f(A_1)$. By hypothesis $A_1 \cong B_0 \otimes_F k$, hence we may show that $\text{Ram}_f(B \otimes_F k) = \text{Ram}_f(B_0 \otimes_F k)$. Suppose first that \mathfrak{P} is a prime of k lying in $\text{Ram}_f(B_0 \otimes_F k)$ and let \mathfrak{p} be the prime of F lying beneath \mathfrak{P} . Then

$$\frac{1}{2} = \text{inv}_{\mathfrak{P}}(B_0 \otimes_F k) = \text{inv}_{\mathfrak{p}}(B_0) \cdot [k_{\mathfrak{P}} : F_{\mathfrak{p}}].$$

By Lemma 3.2 of [8], the local degree $[k_{\mathfrak{P}} : F_{\mathfrak{p}}]$ is odd, hence $\text{inv}_{\mathfrak{p}}(B_0) = \frac{1}{2}$ and $\mathfrak{p} \in \text{Ram}(B_0)$. Since $\text{Ram}(B)$ contains $\text{Ram}(B_0)$, \mathfrak{p} ramifies in B . Then

$$\text{inv}_{\mathfrak{P}}(B \otimes_F k) = \text{inv}_{\mathfrak{p}}(B) \cdot [k_{\mathfrak{P}} : F_{\mathfrak{p}}] = \frac{1}{2} \cdot [k_{\mathfrak{P}} : F_{\mathfrak{p}}]$$

is nontrivial in \mathbb{Q}/\mathbb{Z} . This shows that $\text{Ram}_f(B_0 \otimes_F k) \subseteq \text{Ram}_f(B \otimes_F k)$.

We now show the reverse containment. Suppose that $\mathfrak{Q} \in \text{Ram}_f(B \otimes_F k)$ and let \mathfrak{q} be the prime of F lying beneath \mathfrak{Q} . The argument above shows that $\mathfrak{q} \in \text{Ram}(B)$. If $\mathfrak{q} \in \text{Ram}(B_0)$ then it is necessarily the case that $\mathfrak{Q} \in \text{Ram}_f(B_0 \otimes_F k)$. Assume therefore that $\mathfrak{q} \notin \text{Ram}(B_0)$. Because $\text{Ram}_f(B) = \text{Ram}_f(B_0) \cup \{\nu_1, \nu_2\}$ it must be the case that $\mathfrak{q} \in \{\nu_1, \nu_2\}$. We now obtain a contradiction as

$$\text{inv}_{\mathfrak{Q}}(B \otimes_F k) = \text{inv}_{\mathfrak{p}}(B) \cdot [k_{\mathfrak{Q}} : F_{\mathfrak{p}}] \in \mathbb{Z}$$

since k/F is a Galois extension of degree 2^n , hence the local degree is even if it is not 1 (in which case it corresponds to a prime of F which splits completely in k/F , which contradicts the way we chose ν_1 and ν_2). This shows that $A_1 \cong B \otimes_F k$.

By hypothesis we have $A_2 \cong B \otimes_F k'$. Let ω be a prime of k' lying above ν_1 . By (3), the local invariant $\text{inv}_\omega(A_2)$ is trivial in \mathbb{Q}/\mathbb{Z} , giving us

$$\text{inv}_\omega(A_2) = \text{inv}_\omega(B) \cdot [k'_\omega : F_{\nu_1}] = \frac{1}{2} \cdot [k'_\omega : F_{\nu_1}] \in \mathbb{Z}.$$

This forces the local degree $[k'_\omega : F_{\nu_1}]$ to be even and allows us to conclude that ν_1 does not split completely in k'/F . As the argument that ν_2 does not split completely in k'/F is identical, we have completed the proof of the theorem. \square

Corollary 4.2. *If k and k' are Brauer equivalent number fields then there is no subfield $F \subset k \cap k'$ over which k and k' are Galois with relative degree a power of 2.*

Proof. Let F be a subfield of $k \cap k'$ such that the relative extensions k/F and k'/F are Galois of degree 2^n . Let B_0 be a quaternion division algebra over F and define $A_1 = B_0 \otimes_F k$ and $A_2 = B_0 \otimes_F k'$.

Commutativity of the following diagram implies that $\Phi_{\text{Br}}([A_1]) = [A_2]$:

$$\begin{array}{ccc} \text{Br}(k) & \xleftrightarrow{\Phi_{\text{Br}}} & \text{Br}(k') \\ & \swarrow \text{Res}_k & \nearrow \text{Res}_{k'} \\ & \text{Br}(F) & \end{array}$$

In particular this diagram and the uniqueness of the division algebra representative of a Brauer class shows that if B is a quaternion algebra over F then $A_1 \cong B \otimes_F k$ if and only if $A_2 \cong B \otimes_F k'$. The corollary now follows from Theorem 4.1. \square

Corollary 4.3. *If M_1 and M_2 are incommensurable arithmetic hyperbolic 3-manifolds with Brauer equivalent invariant trace fields k_1 and k_2 then M_1 and M_2 have no proper immersed totally geodesic surfaces in common.*

Proof. Suppose that M_1 and M_2 are as in the statement of the corollary and that they both contain an immersed totally geodesic surface S . It follows from [9, Chapter 9.5] that S is arithmetic and that the invariant trace field of S is a totally real number field F satisfying $[k : F] = 2 = [k' : F]$. Therefore $F = k \cap k'$ and $k/F, k'/F$ are quadratic extensions, hence Galois. Corollary 4.3 now follows from Corollary 4.2. \square

5. MAXIMAL ORDERS

In this section we prove a variety of algebraic results concerning maximal orders in quaternion algebras defined over Brauer equivalent number fields. These results will be applied in Section 6 when we construct incommensurable arithmetic locally symmetric spaces containing exactly the same set of proper immersed totally geodesic surfaces.

Proposition 5.1. *Let K_1 and K_2 be Brauer equivalent number fields and $F \subset K_1 \cap K_2$ a common subfield. Let B_0 be a quaternion algebra over F and define $A_1 = B_0 \otimes_F K_1$ and $A_2 = B_0 \otimes_F K_2$.*

Assume that A_1 and A_2 are division algebras. If B is a quaternion algebra over F then $A_1 \cong B \otimes_F K_1$ if and only if $A_2 \cong B \otimes_F K_2$.

Proof. Suppose that B is a quaternion algebra over F such that $A_1 \cong B \otimes_F K_1$. We will show that $A_2 \cong B \otimes_F K_2$. This suffices to prove the theorem as the same argument, with the roles of A_1 and A_2 reversed will prove the other direction. Because K_1 and K_2 are Brauer equivalent there is an isomorphism $\Phi_{\text{Br}} : \text{Br}(K_1) \rightarrow \text{Br}(K_2)$ which commutes with extension of scalars. It follows that

$$\Phi_{\text{Br}}([A_1]) = \Phi_{\text{Br}}([B_0 \otimes_F K_1]) = [B_0 \otimes_F K_2] = [A_2].$$

The same reasoning implies that

$$[A_2] = \Phi_{\text{Br}}([A_1]) = \Phi_{\text{Br}}([B \otimes_F K_1]) = [B \otimes_F K_2].$$

Since A_2 and $B \otimes_F K_2$ are both quaternion algebras over K_2 , with the former a division algebra, and every Brauer class contains a unique division algebra representative, we conclude that $A_2 \cong B \otimes_F K_2$, as desired. \square

Proposition 5.2. *Let K be a number field, A be a quaternion algebra over K which is unramified at some infinite prime of K and \mathcal{O} be a maximal order in A . If K has narrow class number one then every maximal order in A is conjugate to \mathcal{O} .*

Proof. Section 3 of [6] shows that the number of conjugacy classes of maximal orders in A is equal to the degree over K of a certain class field $K(A)$. This class field is defined as the maximal abelian extension of K which has a 2-elementary Galois group, is unramified outside of the real places of K which ramify in A , and in which all finite primes of K that ramify in A split completely. It follows that $K(A)$ is contained in the narrow class field of K , which by hypothesis is equal to K . The theorem follows. \square

Theorem 5.3. *Let K/F be an extension of number fields, let A be a quaternion algebra over K , B be a quaternion algebra over F and suppose that $A \cong B \otimes_F K$. Identify B with its image in A . If \mathcal{O} is a maximal order of A then $\mathcal{O}_B = \mathcal{O} \cap B$ is a maximal order of B and $\mathcal{O}_B \otimes_{\mathcal{O}_F} \mathcal{O}_K \cong \mathcal{O}$.*

Proof. We begin by showing that $(\mathcal{O} \cap B) \otimes_{\mathcal{O}_F} \mathcal{O}_K \cong \mathcal{O}$. As \mathcal{O} and B are \mathcal{O}_F -modules and \mathcal{O}_K is flat over \mathcal{O}_F we have that tensor products with \mathcal{O}_K commute with (finite) intersections, yielding

$$\begin{aligned} (\mathcal{O} \cap B) \otimes_{\mathcal{O}_F} \mathcal{O}_K &\cong (\mathcal{O} \otimes_{\mathcal{O}_F} \mathcal{O}_K) \cap (B \otimes_{\mathcal{O}_F} \mathcal{O}_K) \\ &\cong \mathcal{O} \cap (B \otimes_F (F \otimes_{\mathcal{O}_F} \mathcal{O}_K)) \\ &\cong \mathcal{O} \cap (B \otimes_F K) \\ &\cong \mathcal{O} \cap A \\ &= \mathcal{O}. \end{aligned}$$

We now show that the order $\mathcal{O} \cap B$ is a maximal order of B . Let Λ be a maximal order of B containing $\mathcal{O} \cap B$. In light of the previous paragraph we see that $\Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_K \cong \mathcal{O}$. In order to show that $\mathcal{O} \cap B = \Lambda$ it suffices to show that the discriminants $d(\mathcal{O} \cap B)$ and $d(\Lambda)$ of these orders are equal (by [16, exer. 3, p. 131]). Exercise 6 of [16, p. 131] shows that because $(\mathcal{O} \cap B) \otimes_{\mathcal{O}_F} \mathcal{O}_K$ and $\Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_K$ are isomorphic, we have that $d(\mathcal{O} \cap B)\mathcal{O}_K = d(\Lambda)\mathcal{O}_K$. This shows that the discriminants of $\mathcal{O} \cap B$ and Λ (these discriminants are ideals of \mathcal{O}_F) extend to the same ideal of \mathcal{O}_K . But different ideals of \mathcal{O}_F must extend to different ideals of \mathcal{O}_K , hence $d(\mathcal{O} \cap B) = d(\Lambda)$ and $\Lambda = \mathcal{O} \cap B$ is maximal. \square

6. SHIMURA VARIETIES CONTAINING THE SAME IMMERSSED TOTALLY GEODESIC SURFACES

Let $K_1 = (\sqrt[8]{-6561})$ and $K_2 = (\sqrt[8]{-16 \cdot 6561})$. It was shown by de Smit and Perlis [4, p. 214] that K_1 and K_2 are non-isomorphic number fields with isomorphic adèle rings, hence by Theorem 1.1 of [7] they are Brauer equivalent. We also note that the only subfield that K_1 and K_2 have in common is \mathbb{Q} . Below we will also make use of the observation that \mathbb{Q} is the only totally real subfield of K_1 and of K_2 . Finally, using the computer algebra system SAGE [17] one can check that both K_1 and K_2 have narrow class number one. As a consequence, Theorem 5.2 implies that in every quaternion algebra over K_1 and K_2 , all maximal orders are conjugate.

Let $A_1 = M_2(K_1)$ and $A_2 = M_2(K_2)$. Denote by \mathcal{O}_{K_1} and \mathcal{O}_{K_2} the rings of integers of K_1 and K_2 and define $\mathcal{O}_1 = M_2(\mathcal{O}_{K_1})$ and $\mathcal{O}_2 = M_2(\mathcal{O}_{K_2})$. These are maximal orders in A_1 and A_2 . Because K_1 and K_2 have signature $(0, 4)$ there are isomorphisms

$$A_1 \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{C})^4 \cong A_2 \otimes_{\mathbb{Q}} \mathbb{R}.$$

Denote by Γ_1 and Γ_2 the images in $SL_2(\mathbb{C})^4$ of $SL_2(\mathcal{O}_{K_1})$ and $SL_2(\mathcal{O}_{K_2})$. The lattices Γ_1 and Γ_2 are arithmetic, non-cocompact and have finite covolume.

Let $G = SL_2(\mathbb{C})^4$, K be a maximal compact subgroup of G and define $M_1 = \Gamma_1 \backslash G / K$ and $M_2 = \Gamma_2 \backslash G / K$. The orbifolds M_1 and M_2 are (incommensurable) arithmetic locally symmetric spaces.

Theorem 6.1. *The arithmetic locally symmetric spaces M_1 and M_2 contain exactly the same set of proper immersed totally geodesic surfaces.*

Proof. Suppose that S is a totally geodesic surface in M_1 and denote by Γ the orbifold fundamental group of S . Then $\Gamma \subset \Gamma_1$. We will show that $\Gamma \subset \Gamma_2$. Let B denote the invariant quaternion algebra of Γ and F denote invariant quaternion algebra of Γ . The field F is a totally real subfield of K_1 , hence by the observation we made above $F = \mathbb{Q}$. As it is necessarily the case that $B \otimes_F K_1 \cong A_1$, we conclude by Theorem 5.3 the intersection $\mathcal{O}_B = \mathcal{O}_1 \cap B$ is a maximal order of B for which $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathcal{O}_{K_1} \cong \mathcal{O}_1$. It follows that $\Gamma \subset \Gamma_{\mathcal{O}_B} \subset \Gamma_1$.

We now show that (up to isomorphism) Γ is contained in Γ_2 . Because K_1 and K_2 are Brauer equivalent and $[B \otimes_F K_1] = [A_1] = [M_2(K_1)]$ is trivial in $\text{Br}(K_1)$, it must be the case that $[B \otimes_F K_2]$ is trivial in $\text{Br}(K_2)$. In particular this shows that $B \otimes_F K_2 \cong M_2(K_2) = A_2$. Identifying B with its image in A_2 we have that the maximal order \mathcal{O}_B is contained in the order $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathcal{O}_{K_2}$ of A_2 . Because every maximal order of K_2 is conjugate (by Theorem 5.2), we conclude that (perhaps upon conjugating \mathcal{O}_B) we have $\mathcal{O}_B \subset \mathcal{O}_2$, hence $\Gamma_{\mathcal{O}_B} \subset \Gamma_2$. As $\Gamma \subset \Gamma_{\mathcal{O}_B}$, this proves that $\Gamma \subset \Gamma_2$. This shows that every proper immersed totally geodesic surface of M_1 is also a proper immersed totally geodesic surface of M_2 . Repeating the same argument with the roles of M_1 and M_2 reversed completes the proof of the theorem. \square

Theorem 6.2. *Let K_1 and K_2 be non-isomorphic Brauer equivalent number fields which are primitive (i.e., have no proper subfields other than \mathbb{Q}). There are infinitely many pairs of incommensurable arithmetic locally symmetric spaces defined over K_1 and K_2 that contain exactly the same set of proper immersed totally geodesic surfaces.*

Proof. We begin by proving that there are infinitely many quaternion division algebras A_1 defined over K_1 satisfying:

- (1) A_1 is indefinite (i.e., some infinite prime of K_1 is unramified in A_1),
- (2) all maximal orders of A_1 are conjugate, and

(3) there exists a quaternion algebra B over \mathbb{Q} such that $B \otimes_{\mathbb{Q}} K_1 \cong A_1$.

Let p_1, p_2 be rational primes which split completely in K_1/\mathbb{Q} and define a quaternion division algebra B over \mathbb{Q} by $\text{Ram}(B) = \{p_1, p_2\}$. Let $A = B \otimes_{\mathbb{Q}} K_1$. If ω is a prime of K_1 lying above one of the p_i then $[(K_1)_{\omega} : \mathbb{Q}_{p_i}] = 1$ (as p_1 splits completely in K_1/\mathbb{Q}), hence

$$\text{inv}_{\omega}(A) = \text{inv}_{\omega}(B \otimes_{\mathbb{Q}} K_1) = [(K_1)_{\omega} : \mathbb{Q}_{p_i}] \cdot \text{inv}_{p_i}(B) = \frac{1}{2} \in \mathbb{Q}/\mathbb{Z},$$

which shows that $\omega \in \text{Ram}(A)$. In particular this shows that A is a division algebra.

As was mentioned in the proof of Proposition 5.2, if A is a quaternion algebra over a number field K which is unramified at an infinite prime of K then Section 3 of [6] implies that the number of conjugacy classes of maximal orders in A is equal to the degree over K of the maximal abelian extension $K(A)$ of K which has a 2-elementary Galois group, is unramified outside of the real places of K which ramify in A , and in which all finite primes of K that ramify in A split completely. This shows that if $K_1(A) = K_1$ then the algebra A satisfies conditions (1)-(3) above.

Suppose therefore that $K_1(A) \neq K_1$ and that A does not satisfy (2) above. Let p_3, p_4 be rational primes (distinct from p_1, p_2) which split completely in K_1/\mathbb{Q} but not in $K_1(A)/\mathbb{Q}$ and define a quaternion algebra B_1 over \mathbb{Q} by $\text{Ram}(B_1) = \{p_1, p_2, p_3, p_4\}$. Let $A_1 = B_1 \otimes_{\mathbb{Q}} K_1$. The definition of $K_1(A)$ and $K_1(A_1)$ implies that $K_1 \subseteq K_1(A_1) \subsetneq K_1(A)$. If $K_1(A_1) = K_1$ then A_1 is the desired quaternion algebra. Otherwise we may continue this construction to obtain a quaternion algebra A_2 with

$$K_1 \subseteq K_1(A_2) \subsetneq K_1(A_1) \subsetneq K_1(A).$$

Because the degree of $K_1(A)$ over K_1 is finite, we may continue this process some finite number of times so as to obtain a quaternion division algebra over K_1 satisfying conditions (1)-(3) above.

The construction above shows that there exists a quaternion algebra A'_1 over K_1 satisfying conditions (1)-(3) above. Similarly, we may construct a quaternion algebra A'_2 over K_2 satisfying properties (1)-(3) as well (with K_2 in place of K_1). If $A'_1 \cong B_1 \otimes_{\mathbb{Q}} K_1$ and $A'_2 \cong B_2 \otimes_{\mathbb{Q}} K_2$ then define a quaternion algebra B over \mathbb{Q} by $\text{Ram}(B) = \text{Ram}(B_1) \cup \text{Ram}(B_2)$. The class field containments used above show that if we define $A_1 := B \otimes_{\mathbb{Q}} K_1$ and $A_2 := B \otimes_{\mathbb{Q}} K_2$ then $K_1(A_1) = K_1$ and $K_2(A_2) = K_2$. In particular all of the maximal orders in A_1 (respectively A_2) are conjugate.

Let (a, b) be the signature of K_1 (and hence of K_2 by Theorem 1.1) so that

$$A_1 \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})^a \times M_2(\mathbb{C})^b \cong A_2 \otimes_{\mathbb{Q}} \mathbb{R}.$$

Let $G = \text{SL}_2(\mathbb{R})^a \times \text{SL}_2(\mathbb{C})^b$ and K be a maximal compact subgroup of G . Let \mathcal{O}_1 be a maximal order of A_1 and \mathcal{O}_2 be a maximal order of A_2 . Finally, define $M_1 = \Gamma_{\mathcal{O}_1} \backslash G/K$ and $M_2 = \Gamma_{\mathcal{O}_2} \backslash G/K$. We will show that M_1 and M_2 contain the same nonempty set of proper immersed totally geodesic surfaces.

Suppose that S is a totally geodesic surface in M_1 and denote by Γ the orbifold fundamental group of S . Then $\Gamma \subset \Gamma_{\mathcal{O}_1}$. We will show that $\Gamma \subset \Gamma_{\mathcal{O}_2}$. Let B_{Γ} denote the invariant quaternion algebra of Γ and F denote invariant trace field of Γ . The field F is a totally real proper subfield of K_1 , hence $F = \mathbb{Q}$ because K_1 is primitive. As it is necessarily the case that $B_{\Gamma} \otimes_F K_1 \cong A_1$, we conclude by Theorem 5.3 the intersection $\mathcal{O}_{B_{\Gamma}} = \mathcal{O}_1 \cap B_{\Gamma}$ is a maximal order of B_{Γ} for which $\mathcal{O}_{B_{\Gamma}} \otimes_{\mathbb{Z}} \mathcal{O}_{K_1} \cong \mathcal{O}_1$. It follows that $\Gamma \subset \Gamma_{\mathcal{O}_{B_{\Gamma}}} \subset \Gamma_{\mathcal{O}_1}$.

We now show that (up to isomorphism) Γ is contained in $\Gamma_{\mathcal{O}_2}$. Proposition 5.1 shows that $B_{\Gamma} \otimes_F K_2 \cong A_2$. Identifying B_{Γ} with its image in A_2 we have that the maximal order $\mathcal{O}_{B_{\Gamma}}$ is contained in the order $\mathcal{O}_{B_{\Gamma}} \otimes_{\mathbb{Z}} \mathcal{O}_{K_2}$ of A_2 . Because every maximal order of K_2 is conjugate

(by Theorem 5.2), we conclude that (perhaps upon conjugating \mathcal{O}_{B_Γ}) we have $\mathcal{O}_{B_\Gamma} \subset \mathcal{O}_2$, hence $\Gamma_{\mathcal{O}_{B_\Gamma}} \subset \Gamma_{\mathcal{O}_2}$. As $\Gamma \subset \Gamma_{\mathcal{O}_{B_\Gamma}}$, this proves that $\Gamma \subset \Gamma_{\mathcal{O}_2}$. This shows that every proper immersed totally geodesic surface of M_1 is also a proper immersed totally geodesic surface of M_2 . Repeating the same argument with the roles of M_1 and M_2 reversed completes the proof of the theorem. \square

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