

PARAMETRIZING SHIMURA SUBVARIETIES OF A_1 SHIMURA VARIETIES AND RELATED GEOMETRIC PROBLEMS

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ABSTRACT. This paper gives a complete parametrization of the commensurability classes of totally geodesic subspaces of irreducible arithmetic quotients of $X_{a,b} = (\mathbf{H}^2)^a \times (\mathbf{H}^3)^b$. A special case describes all Shimura subvarieties of type A_1 Shimura varieties. We produce, for any $n \geq 1$, examples of manifolds/Shimura varieties with precisely n commensurability classes of totally geodesic submanifolds/Shimura subvarieties. This is in stark contrast with the previously studied cases of arithmetic hyperbolic 3-manifolds and quaternionic Shimura surfaces, where the presence of one commensurability class of geodesic submanifolds implies the existence of infinitely many classes.

1. INTRODUCTION

It is well-known that there is a close relationship between irreducible finite-volume quotients of the global symmetric space $X_{a,b} = (\mathbf{H}^2)^a \times (\mathbf{H}^3)^b$, where \mathbf{H}^d denotes d -dimensional hyperbolic space, and quaternion algebras over number fields satisfying certain conditions. For example, see [13]. In this paper we study the parametrization of nonabelian subalgebras of quaternion algebras over number fields and its effects on the geometry of the associated quotient spaces.

Previous work on this problem was restricted to the cases of *arithmetic Kleinian groups* ($a = 0$ and $b = 1$) by Maclachlan–Reid [5] or [7, §9.5] and *quaternionic Shimura surfaces* ($a = 2$ and $b = 0$) by Chinburg and the second author [3]. In both cases the proper closed geodesic subspaces associated with quaternion subalgebras are immersed Riemann surfaces, and one sees that if an irreducible arithmetic quotient of $X_{a,b}$ contains one such geodesic subspace, then it contains infinitely many. Moreover, the collection of all such Riemann surfaces forms infinitely many distinct commensurability classes; recall that two Riemannian manifolds are commensurable if they share an isometric finite-sheeted covering. We will show that this dichotomy of ‘none or infinitely many commensurability classes’ does not necessarily persist for all $X_{a,b}$. For the main technical result, see Theorem 3.1.

First, we describe some positive consequences. For both arithmetic Kleinian groups and quaternionic Shimura surfaces with invariant trace field K , closed geodesic subspaces are associated with quaternion subalgebras defined over quadratic subfields $K_0 \subset K$. In this case K/K_0 is necessarily Galois with Galois group $\mathbb{Z}/2$. We show that this Galois condition is in fact the driving force behind the above dichotomy.

Theorem 4.1. *Let K be a number field and $K_0 \subset K$ a subfield for which K/K_0 is Galois of even degree. If $\Gamma \subset \mathrm{SL}_2(\mathbb{R})^a \times \mathrm{SL}_2(\mathbb{C})^b$ is an irreducible arithmetic lattice arising from a quaternion algebra A over K that contains an irreducible arithmetic sublattice $\Sigma \subset \mathrm{SL}_2(\mathbb{R})^c \times \mathrm{SL}_2(\mathbb{C})^d$ arising from a quaternion algebra B over K_0 , then Γ contains infinitely many pairwise incommensurable arithmetic lattices in $\mathrm{SL}_2(\mathbb{R})^c \times \mathrm{SL}_2(\mathbb{C})^d$.*

For example, this always holds when K/\mathbb{Q} is cyclic of even degree and $A = B \otimes_{\mathbb{Q}} K$ for some \mathbb{Q} -quaternion algebra B (cf. Corollary 4.3). However, such a result does not hold in general. When the extension K/K_0 is either not Galois or is of odd degree, new possible finiteness phenomena arise. For instance in Theorem 4.4 we prove the existence of arithmetic lattices acting on $(\mathbf{H}^2)^6$ containing a unique commensurability class of arithmetic Fuchsian subgroups and no other arithmetic sublattices. In light of our results it seems likely that the ‘nonempty implies infinitely many commensurability classes’ phenomenon is unique to the cases where $X_{a,b}$ has dimension at most

four, i.e., hyperbolic 3-manifolds and quaternionic Shimura surfaces. We note that similar results have been obtained by McReynolds [9].

The following result exemplifies the behavior that can arise when K/K_0 has odd degree.

Theorem 4.5. *Let K/\mathbb{Q} be a totally real cyclic Galois extension of odd degree $n > 1$. There exist infinitely many pairwise incommensurable irreducible lattices $\Gamma \subset \mathrm{SL}_2(\mathbb{R})^n$ with invariant trace field K whose arithmetic sublattices lie in precisely $\tau(n)$ commensurability classes, where τ denotes the divisor function.*

Note that $\tau(n)$ achieves every positive integer value. We emphasize that, even when there are finitely many commensurability classes, each commensurability class determines infinitely many distinct immersed submanifolds. Although this is well-known to experts, we give a proof in Lemma 4.7 for the convenience of the reader.

When K is a totally real field (i.e., $b = 0$) we are considering arithmetic quotients of $(\mathbf{H}^2)^a$. These are also known as *Shimura varieties of type A_1* . In particular, certain special elements of each commensurability class, namely those coming from principal congruence subgroups, parametrize principally polarized abelian varieties with added structure. See [10, p. 87]. Moreover, the Shimura subvarieties are totally geodesic submanifolds of complex codimension $0 < c < a$. The fact that K is totally real makes the classification simpler in this case.

Theorem 1.1. *Let V be a Shimura variety of type A_1 with associated totally real field K and K -quaternion algebra A . Suppose that V is the quotient of $(\mathbf{H}^2)^a$ by an irreducible lattice in $\mathrm{SL}_2(\mathbb{R})^a$, so $\dim_{\mathbb{C}}(V) = a$.*

- (i) *Necessary conditions for V to contain a proper Shimura subvariety of dimension $0 < c < a$ are that $c \mid a$ and for there to exist a proper subfield $K_0 \subset K$ satisfying*

$$[K : K_0] = \frac{a}{c}.$$

- (ii) *Suppose that (i) holds. Then K_0 determines a nonempty set of Shimura subvarieties of V if and only if for every $w \in \mathrm{Ram}(A)$ and v the unique place of K_0 divisible by w , the following hold:*
- (a) *The local degree $[K_w : (K_0)_v]$ is odd.*
 - (b) *If $w' \neq w$ is place of K dividing v and the local degree $[K_{w'} : (K_0)_v]$ is odd then $w' \in \mathrm{Ram}(A)$.*
 - (c) *If every place v' of K_0 is divisible by a place w' of K for which $[K_{w'} : (K_0)_{v'}]$ is odd, then*

$$\{v : v \text{ is a place of } K_0 \text{ divisible by some } w \in \mathrm{Ram}(A)\}$$

has even cardinality.

- (iii) *Suppose that (i) and (ii) hold and that K/K_0 is Galois of even degree. Then V contains infinitely many distinct commensurability classes of distinct Shimura subvarieties arising from K_0 -subalgebras of A .*
- (iv) *Suppose that (i) holds and that K/K_0 has odd degree or is not Galois. Then K_0 could determine any number $0 \leq t \leq \infty$ of distinct commensurability classes of Shimura subvarieties of V . Moreover, every value of t is attained by infinitely many incommensurable Shimura varieties.*

The ingredients for the proof of Theorem 1.1 are a combination of various results below, which we quickly assemble.

Proof of Theorem 1.1. Part (i) follows from Theorem 3.1. See Example 3.2 for details in the totally real case. Part (ii) is Lemma 3.2, and (iii) is Theorem 4.1. The techniques for computing the number t in case (iv) of Theorem 1.1, and examples realizing each value, are described throughout the paper.

For a complete analysis of the odd degree case, one follows the proof of Theorem 4.5 and Lemma 3.2. \square

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2. CONSTRUCTING ARITHMETIC LATTICES

We begin by defining some notation. Throughout this paper K and K_0 will denote number fields with rings of integers \mathcal{O}_K and \mathcal{O}_{K_0} . Given a quaternion algebra B over K , $\text{Ram}(B)$ denotes the set of primes of K , both finite and infinite, that ramify in B . Then $\text{Ram}_\infty(B)$ (resp. $\text{Ram}_f(B)$) denotes the set of archimedean places of K (resp. finite primes of K) that ramify in B . Given a prime $\mathfrak{p} \in \text{Ram}_f(B)$, we denote by $\text{inv}_\mathfrak{p}(B)$ the local Hasse invariant of B at the prime \mathfrak{p} (i.e., the Hasse invariant of the $K_\mathfrak{p}$ -algebra $B \otimes_K K_\mathfrak{p}$).

We now recall the construction of arithmetic lattices acting on products of hyperbolic planes and 3-dimensional hyperbolic spaces. For a more detailed treatment see [2, §3].

Let K be a number field with r_1 real places and r_2 complex places, so $[K : \mathbb{Q}] = r_1 + 2r_2$, and B be a quaternion algebra over K that is not totally definite. The isomorphism

$$B \otimes_{\mathbb{Q}} \mathbb{R} \cong \text{M}_2(\mathbb{R})^s \times \mathbb{H}^r \times \text{M}_2(\mathbb{C})^{r_2},$$

where $s + r = r_1$, determines an embedding

$$\pi : B^\times \hookrightarrow \prod_{v_i \notin \text{Ram}_\infty(B)} (B \otimes_K K_{v_i})^\times \longrightarrow \text{GL}_2(\mathbb{R})^s \times \text{GL}_2(\mathbb{C})^{r_2}.$$

Restricting to the elements B^1 of B^\times with reduced norm 1 induces an embedding

$$\pi : B^1 \hookrightarrow \text{SL}_2(\mathbb{R})^s \times \text{SL}_2(\mathbb{C})^{r_2}.$$

Let \mathcal{O} be a maximal order of B and \mathcal{O}^1 be the multiplicative subgroup of \mathcal{O}^\times consisting of those elements with reduced norm 1. The image $\pi(\mathcal{O}^1) \subset \text{SL}_2(\mathbb{R})^s \times \text{SL}_2(\mathbb{C})^{r_2}$ is a lattice by the work of Borel and Harish-Chandra [1]. An irreducible lattice $\Gamma \subset \text{SL}_2(\mathbb{R})^a \times \text{SL}_2(\mathbb{C})^b$ is arithmetic if and only if it is commensurable with a lattice of the form $\pi(\mathcal{O}^1)$. When $a + b > 1$, all irreducible lattices are in fact arithmetic [8, p. 2]. We note that two arithmetic lattices are commensurable if and only if they are associated with the same number field K and quaternion algebras over K that are $\text{Aut}(K/\mathbb{Q})$ -isomorphic [7, Theorem 8.4.7]. Lastly, an arithmetic lattice is cocompact if and only if its associated quaternion algebra is not isomorphic to the matrix algebra $\text{M}_2(K)$.

3. LATTICES ACTING ON PRODUCTS OF HYPERBOLIC SPACES

Before stating this section’s main theorem we introduce convenient notation for dealing with extensions of archimedean places of number fields. Let K_0 be a number field and v an archimedean place of K_0 . Given a finite degree extension K of K_0 , we define $r_K(v)$ to be the number of extensions of v to K that are real and $c_K(v)$ to be the number of extensions of v to K that are complex. Thus $[K : K_0] = r_K(v) + 2c_K(v)$. As an example, if K is a cubic extension of \mathbb{Q} whose minimal polynomial has exactly one real root and v_∞ denotes the real place of \mathbb{Q} , then $r_K(v_\infty) = 1$ and $c_K(v_\infty) = 1$.

Theorem 3.1. *Let $\Gamma \subset \text{SL}_2(\mathbb{R})^a \times \text{SL}_2(\mathbb{C})^b$ be an irreducible arithmetic lattice arising from a quaternion algebra A over a number field K . There is a one-to-one correspondence between*

- (i) *commensurability classes of irreducible arithmetic lattices of $\mathrm{SL}_2(\mathbb{R})^c \times \mathrm{SL}_2(\mathbb{C})^d$ that are contained in Γ , and*
- (ii) *subfields K_0 of K having precisely d complex places and $\mathrm{Aut}(K_0/\mathbb{Q})$ -isomorphism classes of K_0 -subalgebras $B \subset A$ such that $A \cong B \otimes_{K_0} K$, where B is split at precisely c real places of K_0 .*

Furthermore, under this correspondence it is necessarily the case that

$$[K : K_0] = \frac{2b + a + \sum_{v_j \in \mathrm{Ram}_\infty(B)} r_K(v_j)}{2d + c + \#\mathrm{Ram}_\infty(B)}.$$

Remark. When $a + b > 1$ it follows from the Margulis arithmeticity theorem [8, p. 2] that all irreducible lattices in $\mathrm{SL}_2(\mathbb{R})^a \times \mathrm{SL}_2(\mathbb{C})^b$ are arithmetic, hence the word ‘‘arithmetic’’ may be removed from the first statement in Theorem 3.1.

Before proving Theorem 3.1 we consider a few examples in which Theorem 3.1 simplifies considerably.

Example 3.1. Consider the case in which $(a, b) = (0, 1)$ and $(c, d) = (1, 0)$. In this case Theorem 3.1 provides a criterion for the existence of arithmetic Fuchsian subgroups of a fixed arithmetic Kleinian group. Consider a real place v of K_0 that ramifies in B . If v extends to a complex place of K then the fact that K must have a unique complex place implies that any real place of K_0 splitting in B extends to at least two real places of K which necessarily split in A . This contradicts that fact that A is ramified at all real places of K (implied by our hypothesis that $a = 0$). We therefore deduce that $r_K(v_j) = [K : K_0]$ for all $v_j \in \mathrm{Ram}_\infty(B)$, hence the above formula for $[K : K_0]$ shows that $[K : K_0] = 2$. Theorem 3.1 therefore shows that commensurability classes of arithmetic Fuchsian subgroups of Γ correspond to degree 2 totally real subfields K_0 of K and isomorphism classes of quaternion algebras B over K_0 , split at a unique real place, satisfying $A \cong B \otimes_{K_0} K$. This is Theorem 9.5.4 of [7].

Example 3.2. Now consider the case in which $(a, b) = (n, 0)$ and $(c, d) = (m, 0)$, where $n > m \geq 1$. In this situation both K and K_0 are totally real fields. It follows that every real place of K_0 extends to $[K : K_0]$ real places of K so that $r_K(v_j) = [K : K_0]$ for all $v_j \in \mathrm{Ram}_\infty(B)$. Thus $[K : K_0] = \frac{n}{m}$. In particular this implies that if an irreducible lattice in $\mathrm{SL}_2(\mathbb{R})^n$ contains an irreducible arithmetic lattice in $\mathrm{SL}_2(\mathbb{R})^m$ then $m \mid n$ and $n \geq 2m$. When $n = 2$ and $m = 1$ this reproves Theorem 3.1 of [3].

We now prove Theorem 3.1.

Proof of Theorem 3.1. That a subfield K_0 of K and quaternion algebra B as in (ii) will produce a commensurability class of arithmetic subgroups of $\mathrm{SL}_2(\mathbb{R})^c \times \mathrm{SL}_2(\mathbb{C})^d$ contained in Γ is clear.

Conversely, suppose that Σ is an irreducible arithmetic lattice of $\mathrm{SL}_2(\mathbb{R})^c \times \mathrm{SL}_2(\mathbb{C})^d$ contained in Γ . Denote by B the quaternion algebra associated to Σ and by K_0 the center of B . That K_0 has exactly d complex places is clear. The inclusion $\Sigma \subset \Gamma$ induces an embedding of algebras $B \hookrightarrow A$, and it follows immediately that $A \cong B \otimes_{K_0} K$ and that B splits at precisely c real places of K_0 .

All that remains is to prove that

$$(1) \quad [K : K_0] = \frac{2b + a + \sum_{v_j \in \mathrm{Ram}_\infty(B)} r_K(v_j)}{2d + c + \#\mathrm{Ram}_\infty(B)}.$$

Neither a complex place of K_0 nor a real place of K_0 ramifying in B can extend to a real place of K that is split in A . Indeed, in the latter case we would obtain an induced embedding of Hamilton’s quaternion algebra into $M_2(\mathbb{R})$, which is a contradiction. We conclude that only real places of K_0

that split in B can extend to real places of K splitting in A . Therefore

$$(2) \quad a = \sum_{v_i \notin \text{Ram}_\infty(B)} r_K(v_i)$$

$$(3) \quad b = d[K : K_0] + \sum_{v_i} c_K(v_i),$$

where the latter sum is taken over the real places of K_0 . Formula (1) now follows from combining the equality $[K : K_0] = r_K(v_i) + 2c_K(v_i)$ with equation (3) and substituting (2) into the resulting expression. \square

In concrete applications, one needs a criterion for a K_0 -quaternion algebra B to embed in a K -algebra A . The following lemma gives necessary and sufficient criteria for B to embed in A . This generalizes Theorem 9.5.5 of [7], which considers the case in which $[K : K_0] = 2$.

Lemma 3.2. *Let A be a K -quaternion algebra and $K_0 \subset K$ a proper subfield. Then there exists a K_0 -quaternion algebra B such that $A \cong B \otimes_{K_0} K$ if and only if for every $w \in \text{Ram}(A)$ and v the unique place of K_0 divisible by w , the following conditions hold.*

- (i) *The local degree $[K_w : (K_0)_v]$ is odd.*
- (ii) *If $w' \neq w$ is place of K dividing v and the local degree $[K_{w'} : (K_0)_v]$ is odd then $w' \in \text{Ram}(A)$.*
- (iii) *If every place v' of K_0 is divisible by a place w' of K for which $[K_{w'} : (K_0)_{v'}]$ is odd, then*

$$\{v : v \text{ is a place of } K_0 \text{ divisible by some } w \in \text{Ram}(A)\}$$
has even cardinality.

Proof. Suppose first that B is a K_0 -quaternion algebra such that $A \cong B \otimes_{K_0} K$. Let $w \in \text{Ram}(A)$ and v the unique place of K_0 divisible by w . By [12, Thm. 31.9] we see that

$$\frac{1}{2} = \text{inv}_w(A) = \text{inv}_w(B \otimes_{K_0} K) = [K_w : (K_0)_v] \cdot \text{inv}_v(B) \in \mathbb{Q}/\mathbb{Z},$$

hence we conclude that the local degree $[K_w : (K_0)_v]$ is odd and that B is ramified at v . Now let w' be another place of K dividing v and assume that $[K_{w'} : (K_0)_v]$ is odd. Then

$$\text{inv}_{w'}(A) = \text{inv}_{w'}(B \otimes_{K_0} K) = [K_{w'} : (K_0)_v] \cdot \text{inv}_v(B) = \text{inv}_v(B) = \frac{1}{2} \in \mathbb{Q}/\mathbb{Z},$$

since v ramifies in B . This shows that $w' \in \text{Ram}(A)$.

We now show that (iii) holds. To that end, suppose that every place v' of K_0 is divisible by some place w' of K with $[K_{w'} : (K_0)_{v'}]$ odd. The arguments in the previous paragraph show that

$$\text{Ram}(B) \supseteq \{v : v \text{ is divisible by some } w \in \text{Ram}(A)\}.$$

We will show that this is in fact an equality. To see this, let v' be a place of K_0 that is in $\text{Ram}(B)$ but is not divisible by a place of K ramifying in A . By assumption there is a place w' of K which divides v' and for which $[K_{w'} : (K_0)_{v'}]$ is odd. Observe that $\text{inv}_{w'}(B \otimes_{K_0} K) = [K_{w'} : (K_0)_{v'}] \cdot \text{inv}_{v'}(B)$. Since $w' \notin \text{Ram}(A)$ we necessarily have that $[K_{w'} : (K_0)_{v'}]$ is even, which is a contradiction.

Conversely, suppose that w, w' and v are as in the statement of the lemma and that conditions (i), (ii) and (iii) are satisfied. Let B be the quaternion algebra over K_0 for which

$$\text{Ram}(B) = \{v : v \text{ is divisible by some } w \in \text{Ram}(A)\} \cup S,$$

where the set S is empty if $\{v : v \text{ is divisible by some } w \in \text{Ram}(A)\}$ has even cardinality and consists of a single place v' of K_0 for which every place w' of K dividing v' satisfies $[K_{w'} : (K_0)_{v'}]$ even otherwise. We note that such an algebra B exists because condition (iii) implies that such a v' exists whenever S is required to be non-empty in order to have a set

$$\{v : v \text{ is divisible by some } w \in \text{Ram}(A)\} \cup S$$

of even cardinality.

To show that $A \cong B \otimes_{K_0} K$ we must show that $\text{Ram}(A) = \text{Ram}(B \otimes_{K_0} K)$ (see e.g., [12, §31] and [12, p. 277]). Suppose first that $w \in \text{Ram}(A)$ and let v be the place of K_0 divisible by w . Then

$$\text{inv}_w(B \otimes_{K_0} K) = [K_w : (K_0)_v] \cdot \text{inv}_v(B) = \frac{1}{2} \in \mathbb{Q}/\mathbb{Z}$$

by condition (i) and the fact that $v \in \text{Ram}(B)$. Thus $w \in \text{Ram}(B \otimes_{K_0} K)$.

Now suppose that $w' \in \text{Ram}(B \otimes_{K_0} K)$ and that v is the place of K_0 divisible by w' . Then $\text{inv}_{w'}(B \otimes_{K_0} K) = [K_{w'} : (K_0)_v] \text{inv}_v(B)$, hence B is ramified at v and $[K_{w'} : (K_0)_v]$ is odd. By definition of $\text{Ram}(B)$, v is divisible by some place w of K lying in $\text{Ram}(A)$. We now conclude that $w' \in \text{Ram}(A)$ from condition (ii). This shows that $\text{Ram}(A) = \text{Ram}(B \otimes_{K_0} K)$ and finishes the proof. \square

4. COMMENSURABILITY CLASSES OF ARITHMETIC SUBLATTICES

It is well-known that if an arithmetic hyperbolic 3-manifold contains one totally geodesic surface, then it contains infinitely many distinct commensurability classes of totally geodesic surfaces. In [3, Theorem 1.1] it was shown that if a Shimura surface contains one geodesic curve, then it contains infinitely many that are pairwise incommensurable. The following result gives the appropriate extension to higher rank arithmetic lattices, showing that this phenomenon is closely related to the fact that, in both cases, submanifolds are associated with Galois extensions of even degree.

Theorem 4.1. *Let K be a number field and $K_0 \subset K$ a subfield for which K/K_0 is Galois of even degree. If $\Gamma \subset \text{SL}_2(\mathbb{R})^a \times \text{SL}_2(\mathbb{C})^b$ is an irreducible arithmetic lattice arising from a quaternion algebra A over K that contains an irreducible arithmetic sublattice $\Sigma \subset \text{SL}_2(\mathbb{R})^c \times \text{SL}_2(\mathbb{C})^d$ arising from a quaternion algebra B over K_0 , then Γ contains infinitely many pairwise incommensurable arithmetic lattices in $\text{SL}_2(\mathbb{R})^c \times \text{SL}_2(\mathbb{C})^d$.*

Before proving this, we need the following lemma.

Lemma 4.2. *Let K/K_0 be a finite Galois extension of number fields with even degree. There exist infinitely many places \mathfrak{p} of K_0 that are unramified in K such that the local extension $K_{\mathfrak{q}}/(K_0)_{\mathfrak{p}}$ has even degree for every prime \mathfrak{q} of K over \mathfrak{p} .*

Proof. Let $G = \text{Gal}(K/K_0)$. Since G has even order, it contains an element σ of order two. It follows from the Chebotarev density theorem that there are infinitely many primes \mathfrak{p} of K_0 such that every prime \mathfrak{q} of K above \mathfrak{p} has Frobenius element a conjugate of σ . Since only finitely many primes of K_0 ramify in K , we can always assume that \mathfrak{p} is unramified. With these assumptions we have $[K_{\mathfrak{q}} : (K_0)_{\mathfrak{p}}] = 2$ for every \mathfrak{q} over \mathfrak{p} . The lemma follows. \square

We now prove Theorem 4.1.

Proof of Theorem 4.1. Let Σ, Γ be as in the statement of the theorem. It follows from the theory developed in [7] (see also the discussion in the proof of [3, Thm. 3.1]) that the inclusion $\Sigma \subset \Gamma$ induces an embedding of quaternion algebras $B \hookrightarrow A$. It follows that $A \cong B \otimes_{K_0} K$. By Theorem 3.1, it suffices to show that there exist infinitely many quaternion algebras B' over K_0 , that are pairwise $\text{Aut}(K_0/\mathbb{Q})$ -non-conjugate, have the same archimedean ramification as B , and all satisfy $A \cong B' \otimes_{K_0} K$.

By Lemma 4.2, there are infinitely many primes \mathfrak{p} of K_0 for which $[K_{\mathfrak{q}} : (K_0)_{\mathfrak{p}}]$ is even degree for every prime ideal \mathfrak{q} of K dividing \mathfrak{p} and \mathfrak{p} is totally unramified in K . Pick two such primes \mathfrak{p}_1 and \mathfrak{p}_2 of K_0 .

Let B' be the quaternion algebra over K_0 for which $\text{Ram}(B') = \text{Ram}(B) \cup \{\mathfrak{p}_1, \mathfrak{p}_2\}$. Note that by Reiner [12, Thm. 31.9], the Hasse invariant of $B' \otimes_{K_0} K$ at a prime \mathfrak{q}_i of K lying above \mathfrak{p}_i satisfies

$$\text{inv}_{\mathfrak{q}_i}(B' \otimes_{K_0} K) = [K_{\mathfrak{q}_i} : (K_0)_{\mathfrak{p}_i}] \text{inv}_{\mathfrak{p}_i}(B') = 1 \in \mathbb{Q}/\mathbb{Z},$$

since $[K_{\mathfrak{q}_i} : (K_0)_{\mathfrak{p}_i}]$ is even. This shows that $B' \otimes_{K_0} K$ and $B \otimes_{K_0} K$ are ramified at precisely the same places of K , and hence are both isomorphic to A . There are infinitely many possibilities for $\mathfrak{p}_1, \mathfrak{p}_2$ and hence for B' , which completes the proof of the theorem. \square

The following is an easy consequence of Theorems 3.1 and 4.1.

Corollary 4.3. *If K is an abelian extension of \mathbb{Q} and $\Gamma \subset \text{SL}_2(\mathbb{R})^{2n}$ is an irreducible lattice arising from a quaternion algebra over K and that contains an arithmetic sublattice $\Sigma \subset \text{SL}_2(\mathbb{R})^m$ for some $m|n$, then Γ contains infinitely many pairwise incommensurable arithmetic sublattices arising from some embedding of $\text{SL}_2(\mathbb{R})^m$ into $\text{SL}_2(\mathbb{R})^{2n}$.*

We conclude this section by showing that the phenomenon described above, in which an arithmetic lattice containing a single commensurability class of arithmetic sublattices with positive codimension in fact contains infinitely many, is not true in general. We first study the case where K/K_0 is even degree, but not Galois.

Theorem 4.4. *There exist infinitely many commensurability classes of irreducible arithmetic lattices acting on $(\mathbf{H}^2)^6$ containing exactly one commensurability class of arithmetic Fuchsian subgroups and no other arithmetic sublattices.*

Proof. Let K be the field with minimal polynomial

$$f(t) = t^6 - 10t^4 + 7t^3 + 15t^2 - 14t + 3.$$

This is a totally real sextic extension of \mathbb{Q} with Galois group A_5 . If L/\mathbb{Q} is the Galois closure, then

$$\begin{aligned} \text{Gal}(L/\mathbb{Q}) &\cong A_5 \\ \text{Gal}(L/K) &\cong D_5. \end{aligned}$$

Let p be a rational prime that is unramified in K , and $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the primes of K above p . Considering the conjugacy classes of elements in A_5 and the intersection of each conjugacy class with D_5 , we only have the following possibilities for the local completions:

(i) $r = 6$ and

$$[K_{\mathfrak{p}_i} : \mathbb{Q}_p] = 1 \quad i = 1, \dots, 6$$

(ii) $r = 4$ and

$$\begin{aligned} [K_{\mathfrak{p}_i} : \mathbb{Q}_p] &= 1 & i = 1, 2 \\ [K_{\mathfrak{p}_i} : \mathbb{Q}_p] &= 2 & i = 3, 4 \end{aligned}$$

(iii) $r = 2$ and

$$\begin{aligned} [K_{\mathfrak{p}_1} : \mathbb{Q}_p] &= 1 \\ [K_{\mathfrak{p}_2} : \mathbb{Q}_p] &= 5 \end{aligned}$$

(iv) $r = 2$ and

$$[K_{\mathfrak{p}_i} : \mathbb{Q}_p] = 3 \quad i = 1, 2$$

In particular, we see that for every rational prime p that is unramified in K , there exists a prime \mathfrak{p} of K above p with $[K_{\mathfrak{p}} : \mathbb{Q}_p]$ of odd degree. Recall from the proof of Theorem 4.1 that if B is a quaternion algebra over \mathbb{Q} and $A = B \otimes_{\mathbb{Q}} K$, then

$$\text{inv}_{\mathfrak{p}}(A) = [K_{\mathfrak{p}} : \mathbb{Q}_p] \text{inv}_p(B) \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}.$$

Consequently, if $p \in \text{Ram}(B)$ is a rational prime that is unramified in K , then there exists a prime \mathfrak{p} of K above p such that A ramifies at \mathfrak{p} .

Given a finite set \mathcal{S} of rational primes with even cardinality, containing no primes that ramify in K , let $B_{\mathcal{S}}$ denote the unique \mathbb{Q} -quaternion algebra that ramifies at precisely the places in \mathcal{S} and

$$A_{\mathcal{S}} = B_{\mathcal{S}} \otimes_{\mathbb{Q}} K.$$

Notice that the pair $(\mathbb{Q}, B_{\mathcal{S}})$ determines a commensurability class of arithmetic Fuchsian groups.

Suppose that B' is another \mathbb{Q} -quaternion algebra such that $A_{\mathcal{S}} \cong B' \otimes_{\mathbb{Q}} K$. Then it is not hard to show that $\mathcal{S} \subseteq \text{Ram}(B')$. Indeed, every prime of K at which $A_{\mathcal{S}}$ ramifies lies above a prime in \mathcal{S} . It is also not hard to see that the infinite place v_{∞} of \mathbb{Q} cannot be in $\text{Ram}(B')$, since K is totally real and $A_{\mathcal{S}}$ is unramified at all the real places of K . In particular, we see that if $p \in \text{Ram}(B') \setminus \mathcal{S}$, then p ramifies in K .

With the choice of $f(t)$ as above, the rational primes that ramify in K are 19 and 293. Since $\text{Ram}(B')$ has even cardinality, we see that either $\text{Ram}(B') = \mathcal{S}$, hence $B \cong B'$, or

$$\text{Ram}(B') = \mathcal{S} \cup \{19, 293\}.$$

However, there is a prime \mathfrak{p} of K above 293 of degree 1, i.e., such that $K_{\mathfrak{p}} \cong \mathbb{Q}_{293}$, and it follows that $B' \otimes_{\mathbb{Q}} K$ also ramifies at \mathfrak{p} , which is a contradiction.

To recap, let K be the totally real sextic extension of \mathbb{Q} with minimal polynomial $f(t)$ as above and \mathcal{S} any finite set of rational primes of even cardinality not containing 19 or 293. Let $B_{\mathcal{S}}$ be the unique \mathbb{Q} -quaternion algebra ramified exactly at the primes in \mathcal{S} , and $A_{\mathcal{S}}$ be the base change of $B_{\mathcal{S}}$ from \mathbb{Q} to K . We have shown

$$A_{\mathcal{S}} \cong B' \otimes_{\mathbb{Q}} K$$

for some \mathbb{Q} -quaternion algebra B' if and only if $B' \cong B_{\mathcal{S}}$.

Then $A_{\mathcal{S}}$ determines a commensurability class of arithmetic lattices in $\text{SL}_2(\mathbb{R})^6$, and the above proves that it contains a unique commensurability class of arithmetic Fuchsian subgroups. To see that there are no other arithmetic sublattices, it suffices to notice that K contains no proper subfield other than \mathbb{Q} . This completes the proof. \square

We now study the odd-degree cyclic Galois case.

Theorem 4.5. *Let K/\mathbb{Q} be a totally real cyclic Galois extension of odd degree $n > 1$. There exist infinitely many pairwise incommensurable irreducible lattices $\Gamma \subset \text{SL}_2(\mathbb{R})^n$ with invariant trace field K whose arithmetic sublattices lie in precisely $\tau(n)$ commensurability classes, where τ denotes the divisor function.*

Proof. Let p, q be distinct primes that split completely in K/\mathbb{Q} , let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be the prime divisors in K of $p\mathcal{O}_K$, and let $\mathfrak{q}_1, \dots, \mathfrak{q}_n$ be the prime divisors in K of $q\mathcal{O}_K$. Consider the quaternion algebra A over K that ramifies precisely at the primes $\mathfrak{p}_1, \dots, \mathfrak{p}_n, \mathfrak{q}_1, \dots, \mathfrak{q}_n$. If \mathcal{O} is a maximal order of A , then $\Gamma = \pi(\mathcal{O}^1)$ is an irreducible lattice in $\text{SL}_2(\mathbb{R})^n$. We will show that the arithmetic sublattices in Γ with positive codimension belong to precisely $\tau(n) - 1$ commensurability classes. Since infinitely many primes split completely in K/\mathbb{Q} , our assertion about the existence of infinitely many incommensurable arithmetic lattices Γ with the requisite properties will follow.

Example 3.2 shows that any positive codimension arithmetic sublattice of Γ lies in $\text{SL}_2(\mathbb{R})^d$ for some divisor d of n . Fix one such divisor d and observe that there is a unique subfield K_0 of K for which $[K : K_0] = \frac{n}{d}$. By Theorem 3.1 it suffices to show that there is a unique quaternion algebra B defined over K_0 such that $A \cong B \otimes_{K_0} K$.

To begin, let B be the quaternion algebra over K_0 that is unramified at all archimedean places of K_0 and in which all prime divisors of $p\mathcal{O}_{K_0}$ and $q\mathcal{O}_{K_0}$ ramify. To show that $B \otimes_{K_0} K$ is isomorphic to A we must show that the only nontrivial Hasse invariants of $B \otimes_{K_0} K$ are associated with the

prime divisors of $p\mathcal{O}_K$ and $q\mathcal{O}_K$. Let \mathfrak{p} be a prime divisor of $p\mathcal{O}_{K_0}$ or $q\mathcal{O}_{K_0}$ and \mathfrak{P} be a prime of K lying above \mathfrak{p} . Since p and q both split completely in K/\mathbb{Q} , it follows that \mathfrak{p} splits completely in K/K_0 , hence $[K_{\mathfrak{P}} : (K_0)_{\mathfrak{p}}] = 1$. We now conclude that

$$\text{inv}_{\mathfrak{P}}(B \otimes_{K_0} K) = [K_{\mathfrak{P}} : (K_0)_{\mathfrak{p}}] \text{inv}_{\mathfrak{p}}(B) = 1 \cdot \frac{1}{2} = \frac{1}{2} \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}.$$

Conversely, let \mathfrak{L} be a prime of K not lying above p or q and let $\mathfrak{l} = \mathfrak{L} \cap K_0$. Then

$$(B \otimes_{K_0} K) \otimes_K K_{\mathfrak{L}} \cong (B \otimes_{K_0} K_{\mathfrak{l}}) \otimes_{K_{\mathfrak{l}}} K_{\mathfrak{L}} \cong M_2(K_{\mathfrak{l}}) \otimes_{K_{\mathfrak{l}}} K_{\mathfrak{L}} \cong M_2(K_{\mathfrak{L}}),$$

showing that $B \otimes_{K_0} K$ is unramified at \mathfrak{L} . This shows that $A \cong B \otimes_{K_0} K$.

Now, suppose that B' is a quaternion algebra over K_0 such that $A \cong B' \otimes_{K_0} K$. No real place of K_0 that ramifies in B' can extend to a real place of K splitting in $B' \otimes_{K_0} K$, as this would induce an embedding of Hamilton's quaternions into $M_2(\mathbb{R})$. Since K (and hence K_0) is totally real and A is split at all archimedean places, it follows that B' is split at all archimedean places of K_0 . If \mathfrak{p} is a prime of K lying above p or q then the above argument shows that B' must be ramified at \mathfrak{p} .

We now show that if \mathfrak{l} is a prime of K_0 not lying above p or q then \mathfrak{l} is unramified in B' . Suppose to the contrary that such a prime \mathfrak{l} ramifies in B' and let \mathfrak{L} be a prime of K lying above \mathfrak{l} . Note that $\text{inv}_{\mathfrak{L}}(A) = 1$. Then

$$1 = \text{inv}_{\mathfrak{L}}(A) = \text{inv}_{\mathfrak{L}}(B' \otimes_{K_0} K) = [K_{\mathfrak{L}} : (K_0)_{\mathfrak{l}}] \text{inv}_{\mathfrak{l}}(B') = [K_{\mathfrak{L}} : (K_0)_{\mathfrak{l}}] \cdot \frac{1}{2},$$

forcing us to deduce that

$$[K_{\mathfrak{L}} : (K_0)_{\mathfrak{l}}] = e(\mathfrak{L}/\mathfrak{l})f(\mathfrak{L}/\mathfrak{l})$$

is even. However, this is impossible as $e(\mathfrak{L}/\mathfrak{l})f(\mathfrak{L}/\mathfrak{l})$ is a divisor of $\frac{n}{d}$ (since K/K_0 is Galois), which by hypothesis must be odd. We conclude that B and B' are ramified at precisely the same primes of K_0 , hence they are isomorphic. It follows that B is the unique quaternion algebra over K_0 for which $A \cong B \otimes_{K_0} K$, finishing the proof. \square

The following is an immediate consequence of Theorem 4.5 and the fact that for every $m \geq 1$ there is an integer $n \geq 1$ such that $\tau(n) = m$.

Corollary 4.6. *For every positive integer $m \geq 1$ there exist infinitely many pairwise incommensurable irreducible lattices containing precisely m commensurability classes of arithmetic sublattices.*

We close with the following, which is well-known to experts.

Lemma 4.7. *Let G be a semisimple Lie group of noncompact type with associated symmetric space X and $Y \subset X$ be an embedded proper totally geodesic subspace associated with the subgroup*

$$\text{Stab}_G(Y) = H \subset G.$$

Suppose that Γ is an arithmetic lattice in G such that $\Lambda = \Gamma \cap H$ is a lattice in H . Then X/Γ contains infinitely many distinct immersed totally geodesic submanifolds of the form Y/Λ' with Λ' a lattice in H commensurable with Λ .

Proof. Suppose that $M = X/\Gamma$ and that $N = Y/\Lambda$ admits an immersion as a proper totally geodesic submanifold of M , where Γ (resp. Λ) is a lattice in G (resp. H) as in the statement of the lemma. Then N is nowhere dense in M , and the lift of N to the universal cover $\widetilde{M} = X$ is the Γ -orbit of Y , which represents a chosen lift. In particular, notice that $Y \cdot \Gamma$ is also nowhere dense in X .

Without loss of generality, we can replace G with its adjoint group. Recall that, since Γ is arithmetic, the Margulis dichotomy [8] then implies that the *commensurator*

$$\mathcal{C}(\Gamma) = \{g \in G : \Gamma \text{ is commensurable with } g\Gamma g^{-1}\}$$

is analytically dense in G . Given $c \in \mathcal{C}(\Gamma)$, define

$$\Lambda_c^0 = c\Lambda c^{-1} \cap \Gamma,$$

which is a lattice in $cHc^{-1} \subset G$. Then $\Lambda_c = \Gamma \cap cHc^{-1}$ is a sublattice of Γ containing Λ_c^0 , and hence determines an immersion of $N_c = c(Y)/\Lambda_c$ as a totally geodesic subspace of M , where $c(Y) = Y \cdot c^{-1}$. By construction, Λ_c is commensurable with Λ . From the density of $\mathcal{C}(\Gamma)$ in G and nowhere density of $Y \cdot \Gamma$ in X , we can choose c such that $c(Y)$ is not contained in $\Gamma \cdot Y$. It follows immediately that N_c determines a geodesic submanifold of M distinct from N . Continuing inductively produces arbitrarily many distinct geodesic subspaces of M , which proves the lemma. \square

Remark. The spaces N_i constructed in the proof of Lemma 4.7 are almost certainly not homeomorphic, and have arbitrarily large volume as $i \rightarrow \infty$ (and hence are likely to be quite far from embedded). For some results along these lines, see [6, 11, 4].

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