BOUNDS FOR ARITHMETIC HYPERBOLIC REFLECTION GROUPS IN DIMENSION 2

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ABSTRACT. The work of Nikulin and Agol, Belolipetsky, Storm, and Whyte shows that only finitely many number fields may serve as the field of definition of an arithmetic hyperbolic reflection group. An important problem posed by Nikulin is to enumerate these fields and their degrees. In this paper we prove that in dimension 2 the degree of these fields is at most 7. More generally we prove that the degree of the field of definition of the quaternion algebra associated to an arithmetic Fuchsian group of genus 0 is at most 7, confirming a conjecture of Long, Maclachlan and Reid. We also obtain upper bounds for the discriminants of these fields of definition, allowing for an enumeration which should be useful for the classification of arithmetic hyperbolic reflection groups.

1. INTRODUCTION

A *hyperbolic reflection group* Γ is a discrete subgroup of the isometry group $\mathrm{Isom}(\mathbf{H}^n)$ of hyperbolic n-space which is generated by reflections in the faces of a hyperbolic polyhedron. We say that Γ is a *maximal hyperbolic reflection group* if there does not exist a reflection group $\Gamma' \subset \mathrm{Isom}(\mathbf{H}^n)$ which properly contains Γ. The study of hyperbolic reflection groups has a long and interesting history which goes back to the 19th century. For an overview of this history see for instance the ICM reports of Vinberg [30] and Nikulin [21] and the references therein.

In this paper we will restrict our attention to hyperbolic reflection groups which are *arithmetic*. The reader interested in a detailed account of arithmetic hyperbolic reflection groups is advised to consult the recent survey of Belolipetsky [8]. In 1967 Vinberg [29] gave a criterion for a hyperbolic reflection group to be arithmetic and introduced the notion of the *ground field* (or *field of definition*) of such a group. Answering a longstanding open question, it was proven by Agol, Belolipetsky, Storm and Whyte [2], and independently by Nikulin [23], that there are only finitely many conjugacy classes of arithmetic maximal hyperbolic reflection groups. This makes the classification of such groups feasible, though in practice the quantitative bounds produced by [2] and [23] are large enough to make such a classification unfeasible at present. We note that improved bounds were obtained by Belolipetsky [5] if one restricts to arithmetic maximal hyperbolic reflection groups which are *congruence*. Also note that partial progress towards a general classification has been made in dimension $n = 2$ by Nikulin [22] (see also Allcock [3] and Mark [19]; for a more detailed discussion of these classification results, see [8, p. 18]).

In light of the difficulty of classifying all arithmetic maximal hyperbolic reflection groups it is an important problem to enumerate the totally real number fields which may serve as the field of definition of such a group. While finiteness of these fields of course follows from the aforementioned results, boundedness of the degrees of these fields had already been proven by Long, Maclachlan and Reid [16] in dimension 2, Agol [1] in dimension 3 and Nikulin [20, 23] in dimension $n \geq 4$. (In [23] Nikulin explained that boundedness in dimensions $4 \leq n \leq 9$ follows from the corresponding result in dimensions 2 and 3.)

We now discuss the best known quantitative bounds on the degree of the field of definition of an arithmetic hyperbolic reflection group. In dimension 2 it was proven by Maclachlan [17] that this degree is at most 11. In dimension 3 Belolipetsky and the author [7] showed that the degree is at most 9, improving upon previous work of Belolipetsky [6]. It follows from work of Nikulin [24] that the degree is at most 25 in all other dimensions. In this paper we produce an improved bound in dimension 2.

Theorem 1.1. *The degree over $\mathbb{Q}$ of the field of definition of an arithmetic hyperbolic reflection group in dimension 2 is at most 7.*

In order to prove Theorem 1.1 we prove a more general result (Theorem 1.2) regarding arithmetic Fuchsian groups of genus 0 from which the theorem immediately follows. This more general result suggests that the bound of 7 in Theorem 1.1 may well be optimal.

In their paper [16] Long, Maclachlan and Reid proved that there are only finitely many conjugacy classes in $\mathrm{PSL}_2(\mathbb{R})$ of maximal arithmetic Fuchsian groups of genus 0. They additionally gave a complete classification in the case that the field of definition (of the associated quaternion algebra) is $\mathbb{Q}$ and gave examples with field of definition having degree $1, 2, \ldots, 7$. Based on the extensive computations they conjectured [16, Conjecture 5.6] that the degree of the field of definition of the quaternion algebra associated to an arithmetic Fuchsian group of genus 0 is at most 7. In this paper we confirm this conjecture.

Theorem 1.2. *A positive integer n is the degree over $\mathbb{Q}$ of the field of definition of the quaternion algebra associated to an arithmetic Fuchsian group of genus 0 if and only if $n \in \{1, 2, \ldots, 7\}$.*

The proof of Theorem 1.2 also provides explicit upper bounds on the discriminant of the field of definition of an arithmetic Fuchsian group of genus 0 for each possible degree $n \in \{1, 2, \ldots, 7\}$ (see Theorem 4.1). In degrees $n \leq 6$ these upper bounds are strong enough that all of the relevant totally real number fields have been enumerated and appear in existing databases of totally real number fields of low degree (see for instance [15] and [32]).

We now give a brief outline of the methods used to prove Theorem 1.2. Broadly speaking, our proof is an instance of what Belolipetsky [8] has termed the *spectral method*. To explain what this means, let Γ be a maximal arithmetic Fuchsian group and $\lambda_1(\Gamma)$ be the first non-zero eigenvalue of the Laplacian of Γ. By combining a lower bound for $\lambda_1(\Gamma)$ arising from deep results of Blomer and Brumley [9] on the Ramanujan conjecture over number fields with an upper bound (due to Zograf [33]) for $\lambda_1(\Gamma) \cdot \mathrm{Area}(\mathbf{H}^2/\Gamma)$ which grows linearly in the genus we obtain an upper bound on the co-area of Γ. This method was used crucially by Long, Maclachlan and Reid in [16], and was generalized to dimension 3 by Agol [1]. Equipped with an upper bound on the co-area of Γ, we turn to Borel's classification of maximal arithmetic Fuchsian groups and his formula for their co-areas [10]. Borel's formula can be difficult to work with directly, so we make extensive use of two lower bounds for this co-area. The first bound is due the author and Voight [14] and is most useful when strong bounds for the class number of the field of definition of Γ are known. The second bound is due to Maclachlan [17] and takes into account the number of conjugacy classes of elements of order 2 in Γ. Of particular relevance is the fact that Maclachlan's bound may be employed in the absence of strong class number bounds. Our proof of Theorem 1.2 makes extensive use of these bounds, as well as the discriminant bounds of Odlyzko [25], along with a refinement due to Poitou [26] which takes into account the presence of primes of small norm.

2. ARITHMETIC FUCHSIAN GROUPS

In this section we will briefly review the definitions and basic properties of orders in quaternion algebras and arithmetic Fuchsian groups. The reader desiring a more thorough treatment is advised to consult [28, 18].

2.1. **Quaternion algebras.** Let k be a number field with ring of integers $\mathcal{O}_k$. A quaternion algebra B over k is a central simple k-algebra of dimension 4. If ν is a place of k then $B_\nu = B \otimes_k k_\nu$ is a quaternion algebra over the local field k_ν. If B_ν is a division algebra then we say that ν *ramifies* in B. Otherwise $B_\nu \cong \mathrm{M}_2(k_\nu)$ and we say that ν *splits* in B. The set of places of k (respectively, finite places of k) which ramify in B is denoted by $\mathrm{Ram}(B)$ (respectively, $\mathrm{Ram}_f(B)$). The set $\mathrm{Ram}(B)$ is a finite set of even cardinality. Conversely, given any finite set S of places of k of even cardinality not containing any complex places there is a unique (up to isomorphism) quaternion algebra B over k such that $\mathrm{Ram}(B) = S$. Note that B is a division algebra if and only if $\mathrm{Ram}(B) \neq \emptyset$.

Let k be a number field and B be a quaternion algebra over k. An *order* of B is a subring which is also a finitely generated $\mathcal{O}_k$-module containing a k-basis of B. An order is said to be *maximal* if it is maximal with respect to inclusion. An order $\mathcal{E} \subset B$ is an *Eichler order* if there are maximal orders $\mathcal{O}, \mathcal{O}' \subset B$ such that $\mathcal{E} = \mathcal{O} \cap \mathcal{O}'$. Let S be the (finite) set of finite primes of ν of k such that $\mathcal{O}_\nu \neq \mathcal{O}'_\nu$. If $\nu \in \mathrm{Ram}_f(B)$ then the division algebra B_ν contains a unique maximal order, hence $S \cap \mathrm{Ram}_f(B) = \emptyset$. It follows that if $\nu \in S$ then $B_\nu \cong M_2(k_\nu)$ and there exists a positive integer n_ν such that $\mathcal{E}_\nu = \mathcal{O}_\nu \cap \mathcal{O}'_\nu$ is conjugate to

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \pmod{\pi_\nu^{n_\nu}} \right\},$$

where π_ν is a uniformizer for k_ν. The *level* $\mathfrak{N}$ of $\mathcal{E}$ is defined to be the ideal

$$\mathfrak{N} = \prod_{\nu \in S} \nu^{n_\nu}.$$

Observe that an Eichler order $\mathcal{E}$ is a maximal order if and only if its level is trivial.

2.2. **Arithmetic Fuchsian groups and their co-areas.**

2.2.1. *Definitions.* Let k be a totally real number field and B be a quaternion algebra over k which is split at a unique real place ν of k. Let $\mathcal{E} \subset B$ be an Eichler order and $\mathcal{E}^1$ be the multiplicative subgroup of $\mathcal{E}^*$ generated by elements of norm 1. Finally, let $\Gamma^1_{\mathcal{E}}$ denote the image in $\mathrm{PSL}_2(\mathbb{R})$ of $\mathcal{E}^1$ under the identification $B_\nu \cong B \otimes_k k_\nu \cong \mathrm{M}_2(\mathbb{R})$. The group $\Gamma^1_{\mathcal{E}}$ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ which has finite co-area and which is co-compact if and only if B is a division algebra (equivalently, B is not isomorphic to $\mathrm{M}_2(\mathbb{Q})$).

A Fuchsian group Γ is said to be *arithmetic* if it is commensurable with a group of the form $\Gamma^1_{\mathcal{E}}$. We will call k the *field of definition* of Γ. Also note that the isomorphism class of B determines a (wide) commensurability class of arithmetic Fuchsian groups in $\mathrm{PSL}_2(\mathbb{R})$.

Let $N(\mathcal{E})$ denote the normalizer in B^* of $\mathcal{E}$ and by $\Gamma'_{\mathcal{E}}$ the image of $N(\mathcal{E})$ in $\mathrm{PGL}_2(\mathbb{R})$. Let $\Gamma_{\mathcal{E}} := \Gamma'_{\mathcal{E}} \cap \mathrm{PSL}_2(\mathbb{R})$. Then $\Gamma_{\mathcal{E}}$ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ which contains $\Gamma^1_{\mathcal{E}}$ as a subgroup of finite index. The following is an important theorem of Borel [10].

Theorem 2.1 (Borel)**.** *Every Fuchsian group in the commensurability class determined by B is conjugate to a subgroup of a group of the form $\Gamma_{\mathcal{E}}$ for some Eichler order $\mathcal{E}$ of square-free level.*

Recall that a subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is a *congruence subgroup* if there is a positive integer N such that Γ contains the *principal congruence subgroup* $\Gamma[N]$ of $\mathrm{SL}_2(\mathbb{Z})$, which is defined as the kernel of the reduction map

$$\mathrm{SL}_2(\mathbb{Z}) \longrightarrow \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}).$$

We now extend the notion of congruence to our cocompact setting.

Let $\mathcal{O}$ be a maximal order contained in B and I be an integral 2-sided $\mathcal{O}$-ideal. The principal congruence subgroup of $\Gamma^1_{\mathcal{O}}$ of level I is defined to be the image in $\mathrm{PSL}_2(\mathbb{R})$ of

$$\mathcal{O}^1(I) = \{\alpha \in \mathcal{O}^1 : \alpha - 1 \in I\}.$$

We will denote this group by $\Gamma^1_{\mathcal{O}}(I)$. With this notation in place, we define an arithmetic Fuchsian group to be *congruence* if it contains a group of the form $\Gamma^1_{\mathcal{O}}(I)$.

In this paper it will be especially important that every maximal arithmetic Fuchsian group (i.e., group of the form $\Gamma_{\mathcal{E}}$) is congruence. This was proven by Long, Maclachlan and Reid [16, Lemma 4.2].

Proposition 2.2 (Long, Maclachlan and Reid)**.** *If $\mathcal{E}$ is an Eichler order then $\Gamma_{\mathcal{E}}$ is a congruence subgroup.*

2.2.2. *Coareas.* Let k be a totally real number field of degree n, B a quaternion algebra over k which is split at precisely one real place of k and $\mathcal{E} \subset B$ be an Eichler order of level $\mathfrak{N}$. Then

$$\mathrm{Area}(\mathbf{H}^2/\Gamma^1_{\mathcal{E}}) = \frac{8\pi\zeta_k(2)d^{3/2}}{(4\pi^2)^n} \prod_{\mathfrak{p}\in \mathrm{Ram}_f(B)} (N\mathfrak{p}-1) \prod_{\mathfrak{q}\mid\mathfrak{N}} (N\mathfrak{q}+1), \tag{2.1}$$

where d is the absolute value of the discriminant of k, $\zeta_k(2)$ is the Dedekind zeta function of k evaluated at $s = 2$ and $N\mathfrak{p}, N\mathfrak{q}$ denote the norms of the prime ideals $\mathfrak{p}, \mathfrak{q} \subset \mathcal{O}_k$.

Let $\mathcal{E}$ be an Eichler order of square-free level $\mathfrak{N}$ and recall that every maximal arithmetic Fuchsian group is of the form $\Gamma_{\mathcal{E}}$. In this paper we will on many occasions need to analyze the index $[\Gamma_{\mathcal{E}} : \Gamma^1_{\mathcal{E}}]$. Although a formula for this index was given by Borel [10], it will be more convenient to make use of the following upper bound (derived by the author and Voight [14, Lemma 1.15 and Prop. 1.17]).

Proposition 2.3 (Linowitz and Voight)**.** *The index $[\Gamma_{\mathcal{E}} : \Gamma^1_{\mathcal{E}}]$ satisfies*

$$[\Gamma_{\mathcal{E}} : \Gamma^1_{\mathcal{E}}] \le 2^{m+r+s+h_2},$$

where $r = \#\,\mathrm{Ram}_f(B)$, s denotes the number of prime divisors of $\mathfrak{N}$, m denotes the rank (over $\mathbb{F}_2$) of the group of totally positive units of $\mathcal{O}_k$ modulo squares, and h_2 denotes the rank (over $\mathbb{F}_2$) of the 2-part of the ideal class group of k. Moreover, if this bound is an equality then every prime lying in $\mathrm{Ram}_f(B)$ belongs to a square class in the strict class group of k.

While the above bound for $[\Gamma_{\mathcal{E}} : \Gamma^1_{\mathcal{E}}]$ is very useful when one has a good upper bound for the 2-part of the class group of k, an alternative bound was proven by Maclachlan [17, Theorem 2.2] and is useful in the absence of strong class number bounds.

Theorem 2.4 (Maclachlan)**.** *The index $[\Gamma_{\mathcal{E}} : \Gamma^1_{\mathcal{E}}]$ is a power of 2 and satisfies*

$$[\Gamma_{\mathcal{E}} : \Gamma^1_{\mathcal{E}}] \le \ell_2 \cdot 2^{r+2s'},$$

where $r = \#\,\mathrm{Ram}_f(B)$, s' is the number of non-dyadic prime divisors of $\mathfrak{N}$ and ℓ_2 is the number of conjugacy classes of elements of order 2 in $\Gamma_{\mathcal{E}}$.

We conclude this section by formally stating the following consequence of Borel's volume formula [10].

Theorem 2.5 (Borel)**.** *Let k be a totally real number field, B a quaternion algebra over k which is split at precisely one real place of k, and $\mathcal{O} \subset B$ a maximal order. Of the arithmetic Fuchsian groups in the commensurability class defined by (k, B), the group $\Gamma_{\mathcal{O}}$ has minimal co-area.*

3. A BOUND ON THE CO-AREA OF A CONGRUENCE ARITHMETIC FUCHSIAN GROUP OF GENUS 0

In this brief section we will derive an upper bound for the co-area of a congruence arithmetic Fuchsian group of genus 0. Our derivation closely follows that of Long, Maclachlan and Reid [16, p. 7], who showed that such a group has co-area at most $\frac{128\pi}{3}$.

Theorem 3.1. *If Γ is a congruence arithmetic Fuchsian group of genus 0 then*

$$\mathrm{Area}(\boldsymbol{H}^2/\Gamma) < 34\pi.$$

Theorem 3.1 is an immediate consequence of the following upper and lower bounds for the first non-zero eigenvalue $\lambda_1(\Gamma)$ of the Laplacian of Γ. We note that the upper bound is due to Zograf [33], while the lower bound follows from the Jacquet-Langlands correspondence together with bounds on the Ramanujan conjecture over number fields due to Blomer and Brumley [9].

Theorem 3.2 (Zograf)**.** *Let Γ be a Fuchsian group of finite co-area and denote by $g(\Gamma)$ the genus of $\boldsymbol{H}^2/\Gamma$. If $\mathrm{Area}(\boldsymbol{H}^2/\Gamma) > 32\pi$ then*

$$\lambda_1(\Gamma) < \frac{8\pi(g(\Gamma)+1)}{\mathrm{Area}(\boldsymbol{H}^2/\Gamma)}.$$

Theorem 3.3 (Blomer and Brumley)**.** *If Γ is a congruence arithmetic Fuchsian group then*

$$\lambda_1(\Gamma) \geq \frac{975}{4096}.$$

4. PROOF OF THEOREM 1.2

In this section we prove Theorem 1.2. As was mentioned in the introduction, Long, Maclachlan and Reid [16] produced, for every $n \in \{1, \dots, 7\}$, an arithmetic Fuchsian group of genus 0 for which the field of definition of the associated quaternion algebra has degree n. Moreover, Maclachlan [17, Theorem 1.1] has shown that if k is the field of definition of an arithmetic Fuchsian group of genus 0 then $[k : \mathbb{Q}] \leq 11$, hence we must show that if $n \in \{8, \dots, 11\}$ and k is a totally real number field of degree n then k is not the defining field of an arithmetic Fuchsian group of genus 0. As our method of proof varies little from case to case we will give a complete proof when $n = 8$ (the most difficult case) and leave the remaining cases to the reader.

Let k be a totally real number field of degree n and B be a quaternion algebra defined over k which is split at precisely one real place of k. Suppose that Γ is an arithmetic Fuchsian group of genus 0 which has invariant trace field k and invariant quaternion algebra B. Let $\mathcal{E} \subset B$ be an Eichler order of square-free level $\mathfrak{N}$ for which Γ is contained in the maximal arithmetic Fuchsian group $\Gamma_{\mathcal{E}}$. Then $\Gamma_{\mathcal{E}}$ must also have genus 0 and is congruence by Proposition 2.2. We now conclude from Theorem 3.1 that $\mathrm{Area}(\mathbf{H}^2/\Gamma_{\mathcal{E}}) < 34\pi$.

Let $\Gamma_{\mathcal{E}}$ have signature $(0; m_1, \ldots, m_t)$ so that

$$34\pi > \text{Area}(\mathbf{H}^2/\Gamma_{\mathcal{E}}) = 2\pi \left(-2 + \sum_{i=1}^{t} (1 - \frac{1}{m_i}) \right),$$

hence $\ell_2(S) \leq 38$. We now conclude from Theorem 2.4, equation (2.1), and the trivial bound $\zeta_k(2) \geq 1$ that the root discriminant $\delta := d^{1/n}$ of k satisfies $\delta < 27.716$.

We will henceforth assume that $n = 8$. We now employ a refinement of the Odlyzko discriminant bounds in order to show that the class number h of k satisfies $h \leq 3$. These bounds were originally developed by Odlyzko [25], though they were later refined by Poitou [26] so as to take into account the existence of primes of small norm. The bounds were subsequently refined again, by Brueggeman and Doud [12]. In particular, the precise formula for the Odlyzko-Poitou bounds which we will employ throughout the remainder of this paper is the one appearing in [12, Theorem 2.4]. This formula may easily be computed using a computer algebra system like SAGE [27].

If the class number h of k satisfies $h \geq 4$ then the Hilbert class field H of k is a totally real field of degree at least 32 and root discriminant $\delta < 27.716$. The Odlyzko-Poitou bounds imply that the root discriminant of H must be at least 28.111, a contradiction which proves that $h \leq 3$. Denote by h_2 the rank over $\mathbb{F}_2$ of the 2-part of the ideal class group of k. As $2^{h_2} \leq h$ we deduce that $h_2 \leq 1$. In fact we will show that $h_2 = 0$. To prove this we begin by letting w_2 denote the number of primes of $\mathcal{O}_k$ having norm 2. Then $w_2 \leq 8$, and observing the Euler product expansion of $\zeta_k(2)$ we see that $\zeta_k(2) \geq (4/3)^{w_2}$. Our bound $\text{Area}(\mathbf{H}^2/\Gamma_{\mathcal{E}}) < 34\pi$, along with Theorem 2.4 and equation (2.1), now implies that

$$\delta < 17.463 \cdot 1.036^{w_2}. \tag{4.1}$$

Recall that we have shown that $h_2 \leq 1$. Suppose that $h_2 = 1$. As $2^{h_2} \mid h$ and $h \leq 3$ we see that $h = 2$. Thus the Hilbert class field H of k (which must be totally real and have root discriminant δ) is a quadratic extension of k. Moreover, the w_2 primes of norm 2 in k either split completely or else are inert in H/k. In particular H has at least w_2 primes with norm at most 4. The table below gives the numerical bound for δ corresponding to each value of w_2 in $\{0, \ldots, 8\}$ that one gets from equation (4.1), along with the Odlyzko-Poitou bound for the root discriminant of a totally real field of degree 16 and at least w_2 primes of norm at most 4.

(4.2)

w_2	$\delta <$	Odlyzko-Poitou
0	17.464	18.731
1	18.092	20.024
2	18.743	21.407
3	19.418	22.885
4	20.117	24.465
5	20.841	26.154
6	21.592	28.764
7	22.369	31.038
8	23.174	33.455

From (4.2) we see that for each value of w_2 the upper bound for δ which arises from the inequality $\delta < 17.463 \cdot 1.036^{w_2}$ is strictly less than the lower bound for δ which we obtain from the Odlyzko-Poitou root discriminant bounds. This contradiction shows that $h_2 \neq 1$, hence $h_2 = 0$ as claimed.

Let m denote the rank over $\mathbb{F}_2$ of the group of totally positive units of $\mathcal{O}_k$ modulo squares. We will now employ the upper bound for the index $[\Gamma_{\mathcal{E}} : \Gamma^1_{\mathcal{E}}]$ given in Proposition 2.3. First notice that if $\mathfrak{p}, \mathfrak{q}$ are primes of $\mathcal{O}_k$ with $N\mathfrak{p} > 2$ then we have

$$\frac{N\mathfrak{p}-1}{2}, \frac{N\mathfrak{q}+1}{2} \geq 1,$$

hence by (2.1) and Proposition 2.3 we have

$$\delta < 13.07994 \cdot 1.05947^{m+r_2} \cdot 0.97632^{w_2}, \tag{4.3}$$

where r_2 denotes the number of primes of norm 2 contained in $\mathrm{Ram}_f(B)$. Here we have once again used the bound $\zeta_k(2) \geq (4/3)^{w_2}$.

We will now show that $\delta < 17$. We begin by noting that a theorem of Armitage and Fröhlich [4] shows that $h_2 \geq m - \lfloor n/2 \rfloor = m - 4$, hence $m \leq 4$ as we have already concluded that $h_2 = 0$. Therefore $m \leq 4$ and $r_2 \leq w_2 \leq 8$. (We note that the inequality $r_2 \leq w_2$ follows directly from the definitions of r_2 and w_2 and the bound $w_2 \leq 8$ follows from the fact that a number field of degree 8 has at most 8 primes of norm 2.) Suppose that $m = 4$. If $w_2 \geq 3$ then one may check that the upper bound for δ given by (4.3) is strictly less than the lower bound for δ given by applying the Odlyzko-Poitou bounds (see also the supplementary tables at [13]) to k, a totally real field of degree 8 containing at least w_2 primes of norm 2. Thus $w_2 \leq 2$ and $\delta \leq 17.633$ by (4.3). If $w_2 = 2$ and $r_2 = 1$ then we see that $\delta < 16.644$. If $w_2 = r_2 = 2$, then $\mathrm{Ram}_f(B)$ must contain a prime of norm greater than 2 as B must ramify at an odd number of finite primes (since B is ramified at 7 real places of k and $\#\,\mathrm{Ram}(B)$ is even). If this prime has norm at least 4 then $\delta < 16.643$ by (2.1) and Proposition 2.3. If this prime has norm 3 then the Euler product expansion of $\zeta_k(2)$ gives us the improved bound $\zeta_k(2) \geq (4/3)^{w_2}(9/8)$, which implies that $\delta < 17.460$. The Odlyzko-Poitou bounds imply that a totally real field of degree 8 with two primes of norm 2 and a prime of norm 3 has root discriminant at least 17.470, a contradiction. We conclude that either $w_2 = 2$ and $r_2 = 0$ or else $w_2 \leq 1$. In the former case (4.3) implies that $\delta < 15.709$, hence we may assume $w_2 \leq 1$. Maclachlan [17, p. 115] has shown that if $\mathrm{Ram}_f(B)$ consists of a single prime then k must contain a prime of norm 2 not in $\mathrm{Ram}_f(B)$. This shows that we cannot have $w_2 = r_2 = 1$ with $\mathrm{Ram}_f(B)$ consisting of a single prime of norm 2. An examination of the remaining cases shows that we always have $\delta < 17$. This shows that $\delta < 17$ when $m = 4$. The same arguments (employed in the same manner) imply that $\delta < 17$ in the remaining cases when $m \in \{0, 1, 2, 3\}$.

To conclude, we note that all totally real number fields of degree 8 with root discriminant less than 17 have been enumerated and appear in the LMFDB database [15]. There are 920 such fields. Using the computer algebra system MAGMA [11] we find that none of these fields support a quaternion algebra which is split at a unique real place and which contains a maximal order $\mathcal{O}$ for which $\mathrm{Area}(\mathbf{H}^2/\Gamma_{\mathcal{O}}) < 34\pi$, subject to the caveat that if $\#\,\mathrm{Ram}_f(B) = 1$ then k has a prime of norm 2 not lying in $\mathrm{Ram}_f(B)$ (a necessary condition for the existence of an arithmetic Fuchsian group of genus 0 by Maclachlan [17, p. 115]). In light of Theorem 2.5 and Theorem 3.1 this shows that no totally real field of degree 8 is the field of definition of the quaternion algebra associated to an arithmetic Fuchsian group of genus 0. As was mentioned above, the proofs for degrees 9, 10, 11 are similar (though easier) and left to the reader. We simply remark that in degrees 10 and 11 one additionally uses the fact that no totally real field of degree at least 10 has root discriminant less than 14 (see [31]).

In this manner we also obtain upper bounds on δ for $n \leq 7$. We list these bounds below in (4.4). For the convenience of the reader we also list the bound Δ such that all totally real number fields of degree n with root discriminant less than Δ have been enumerated ([15] and [32]). Note that for $n \leq 6$ our upper bound for δ is always less than the corresponding value of Δ. It follows that in these degrees, the field of definition of an arithmetic Fuchsian group of genus 0 lies in a finite (though extremely long) list of fields which may be obtained from existing databases of number fields of low degree. In these cases ($n \leq 6$) we additionally give the number of totally real fields of degree n with root discriminant less than our upper bound for δ. This information appears in the column marked "# fields".

(4.4)

n	$\delta <$	Δ	# fields
2	78.130	1000	1859
3	45.265	300	4422
4	34.453	100	19298
5	29.249	35	24748
6	26.224	28	45616
7	18.512	15.5	> 301

We summarize this discussion as follows.

Theorem 4.1. *The root discriminant of the field of definition k of the quaternion algebra associated to an arithmetic Fuchsian group of genus 0 is bounded above by the entry corresponding to $n = [k : \mathbb{Q}]$ in (4.4).*

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