NEWFORM THEORY FOR HILBERT EISENSTEIN SERIES

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Abstract. In his thesis, Weisinger [17] developed a newform theory for elliptic modular Eisenstein series. This newform theory for Eisenstein series was later extended to the Hilbert modular setting by Wiles [18]. In this paper we extend the theory of newforms for Hilbert modular Eisenstein series. In particular we provide a strong multiplicity-one theorem in which we prove that Hilbert Eisenstein newforms are uniquely determined by their Hecke eigenvalues for any set of primes having Dirichlet density greater than $\frac{1}{2}$. Additionally, we provide a number of applications of this newform theory. Let $\mathscr{E}_k(\mathcal{N},\Psi)$ denote the space of Hilbert modular Eisenstein series of parallel weight $k \geq 3$, level $\mathcal{N}$ and Hecke character Ψ over a totally real field K. For any prime $\mathfrak{q}$ dividing $\mathcal{N}$, we define an operator $C_\mathfrak{q}$ generalizing the Hecke operator $T_\mathfrak{q}$ and prove a multiplicity-one theorem for $\mathscr{E}_k(\mathcal{N},\Psi)$ with respect to the algebra generated by the Hecke operators $T_\mathfrak{p}$ ($\mathfrak{p} \nmid \mathcal{N}$) and the operators $C_\mathfrak{q}$ ($\mathfrak{q} \mid \mathcal{N}$). We conclude by examining the behavior of Hilbert Eisenstein newforms under twists by Hecke characters, proving a number of results having a flavor similar to those of Atkin and Li [2].

1. Introduction

Let $\mathcal{S}_k(N,\psi)$ be the complex vector space of elliptic modular cusp forms of weight k, level N and Dirichlet character ψ. The theory of newforms [1, 8] shows that $\mathcal{S}_k(N,\psi)$ decomposes into a direct sum of common eigenspaces for the algebra generated by the Hecke operators T_p (for primes $p \nmid N$). The generators of the one-dimensional eigenspaces are called *newforms* of level N. The eigenspaces of dimension greater than one are generated by a newform of level M dividing N and its shifts by the various divisors of NM^{-1}. Of immense importance in the theory of newforms is the multiplicity-one theorem, which states that if f_1 and f_2 are newforms of level N_1 and N_2 and the T_p eigenvalue of f_1 is equal to that of f_2 for almost all primes $p \nmid N_1N_2$ then $N_1 = N_2$ and f_1 is a scalar multiple of f_2.

In his thesis [17] Weisinger considered the space $\mathcal{E}_k(N,\psi)$ of Eisenstein series of weight k, level N and character ψ and showed that $\mathcal{E}_k(N,\psi)$ possesses a newform theory analogous to that of $\mathcal{S}_k(N,\psi)$. A well known construction associates to a pair of Dirichlet characters ψ_1,ψ_2 modulo N_1,N_2 (satisfying $\psi_1(-1)\psi_2(-1) = (-1)^k$) an Eisenstein series $E_{\psi_1,\psi_2} \in \mathcal{E}_k(N_1N_2,\psi_1\psi_2)$. Under this construction the n-th coefficient of $E_{\psi_1\psi_2}$ is equal to $\sum_{d|n}\psi_1(\frac{n}{d})\psi_2(d)d^{k-1}$. Weisinger defined an *Eisenstein newform* to be an Eisenstein series $E_{\psi_1,\psi_2} \in \mathcal{E}_k(N_1N_2,\psi_1\psi_2)$ where ψ_1 and ψ_2 are primitive characters modulo N_1 and N_2. After showing that an Eisenstein newform is a simultaneous eigenform for all Hecke operators T_p

2010 *Mathematics Subject Classification.* Primary 11F41; Secondary 11F11.
Key words and phrases. Hilbert modular form, Eisenstein series, newform.

with p not dividing its level, Weisinger proved a multiplicity-one theorem for Eisenstein newforms: if E_{ψ_1,ψ_2} and E_{χ_1,χ_2} are Eisenstein newforms of levels N and M respectively and the T_p eigenvalue of E_{ψ_1,ψ_2} is equal to that of E_{χ_1,χ_2} for almost all primes $p \nmid NM$ then $N = M$ and E_{ψ_1,ψ_2} is a scalar multiple of E_{χ_1,χ_2}.

The theory of newforms was later extended to spaces of Hilbert modular cusp forms by Shimura [16] and Shemanske-Walling [14]. Let K be a totally real number field and $\mathscr{E}_k(\mathcal{N}, \Psi)$ denote the space of Hilbert Eisenstein series of parallel weight $k \geq 3$, level $\mathcal{N}$ and Hecke character Ψ over K. Wiles [18] was the first to extend Weisinger's theory of Eisenstein newforms to $\mathscr{E}_k(\mathcal{N}, \Psi)$. Let ψ_1, ψ_2 be characters on the ideal class group of K modulo $\mathcal{N}_1\mathfrak{P}_\infty$ and $\mathcal{N}_2\mathfrak{P}_\infty$ respectively. Associated to ψ_1 and ψ_2 is an Eisenstein series $E_{\psi_1,\psi_2} \in \mathscr{E}_k(\mathcal{N}, \Psi)$ where Ψ is the Hecke character such that $\Psi^*(\mathfrak{r}) = \psi_1(\mathfrak{r})\psi_2(\mathfrak{r})$ for all integral ideals $\mathfrak{r}$ coprime to $\mathcal{N}_1\mathcal{N}_2$ (see Section 2 for notation). Wiles declared E_{ψ_1,ψ_2} to be a newform of level $\mathcal{N}$ if both ψ_1 and ψ_2 are primitive and $\mathcal{N} = \operatorname{cond}(\psi_1)\operatorname{cond}(\psi_2)$. He then showed that $\mathscr{E}_k(\mathcal{N}, \Psi)$ is generated by the newforms of level $\mathcal{M}$ dividing $\mathcal{N}$ and their shifts by the divisors of $\mathcal{N}\mathcal{M}^{-1}$.

In this paper we extend Wiles' theory of newforms for Hilbert Eisenstein series and then explore a few applications. We begin by introducing a bit of notation. If $f \in \mathscr{E}_k(\mathcal{N}, \Psi)$ and $\mathfrak{m}$ is an integral ideal then denote by $C(\mathfrak{m}, f)$ the $\mathfrak{m}$-th "Foruier coefficient" of f (see Section 2 for a precise definition). If f is a newform of level $\mathcal{N}$ and $\mathfrak{p}$ is a prime not dividing $\mathcal{N}$ then the $T_\mathfrak{p}$-eigenvalue of f is equal to $C(\mathfrak{p}, f)$ (see Proposition 3.3). In Section 3 we prove a strong multiplicity-one theorem for Hilbert Eisenstein newforms:

Theorem 1.1. *Let $\mathcal{N}, \mathcal{M}$ be integral ideals and E_{ψ_1,ψ_2} and E_{χ_1,χ_2} be newforms of level $\mathcal{N}$ and $\mathcal{M}$ such that*
$$C(\mathfrak{p}, E_{\psi_1,\psi_2}) = C(\mathfrak{p}, E_{\chi_1,\chi_2})$$
for a set of primes having Dirichlet density strictly greater than $\frac{1}{2}$. Then $\mathcal{N} = \mathcal{M}$ and $E_{\psi_1,\psi_2} = E_{\chi_1,\chi_2}$.

Our multiplicity-one theorem is considerably stronger than the multiplicity-one theorems mentioned above. Unlike the previous theorems, which show that newforms are uniquely determined by their Hecke eigenvalues for almost all primes $\mathfrak{p}$, our theorem shows that Eisenstein newforms are uniquely determined by their Hecke eigenvalues for any set of primes with Dirichlet density greater than $\frac{1}{2}$. Furthermore, our theorem is best possible in the sense that it is not difficult to construct distinct newforms whose Hecke eigenvalues agree for a set of primes of density equal to $\frac{1}{2}$. Such a strong multiplicity-one theorem is possible because of the close relationship between the arithmetic of Hilbert Eisenstein newforms and the characters to which they are associated. Indeed, the orthogonality relations for characters on finite abelian groups play a crucial role in the proof of our multiplicity-one theorem. It should be noted that in the context of Hilbert cusp forms, the strongest multiplicity-one theorem known is due to Ramakrishnan [12], who showed that if two newforms have identical Hecke eigenvalues for a set of primes with Dirichlet density greater than $\frac{7}{8}$, then the newforms are scalar multiples of one another.

After extending the theory of newforms for Hilbert modular Eisenstein series in Section 3, we devote the remainder of this paper to the study of three themes.

The first theme that we consider is an extension of the multiplicity-one theorem from the subspace of $\mathscr{E}_K(\mathcal{N},\Psi)$ generated by newforms to all of $\mathscr{E}_k(\mathcal{N},\Psi)$. In Section 4 we define an operator $C_\mathfrak{q}$ (for $\mathfrak{q}$ a prime dividing $\mathcal{N}$) which acts on the space $\mathscr{E}_k(\mathcal{N},\Psi)$. The action of this operator on spaces of cusp forms was studied in [11, 9, 3], where it was shown that the space $\mathscr{S}_k(\mathcal{N},\Psi)$ is a direct sum of common eigenspaces of the algebra generated by the Hecke operators T_p (for $\mathfrak{p}\nmid\mathcal{N}$) and the operators $C_\mathfrak{q}$ (for $\mathfrak{q}\mid\mathcal{N}$), each of dimension one. Furthermore, when restricted to the subspace of $\mathscr{S}_k(\mathcal{N},\Psi)$ generated by newforms of exact level $\mathcal{N}$, the action of $C_\mathfrak{q}$ coincides with that of the Hecke operator $T_\mathfrak{q}$. After reviewing the construction of the $C_\mathfrak{q}$ operator we prove the following multiplicity-one theorem for $\mathscr{E}_k(\mathcal{N},\Psi)$:

Theorem 1.2. *The space $\mathscr{E}_k(\mathcal{N},\Psi)$ can be decomposed into a direct sum of common eigenspaces of the algebra generated by $\{T_\mathfrak{p} : \mathfrak{p}\nmid\mathcal{N}\}$ and $\{C_\mathfrak{q} : \mathfrak{q}\mid\mathcal{N}\}$, each of dimension one.*

Our second theme concerns the behavior of Hilbert Eisenstein series under twists by Hecke characters and is the topic of Section 5. This section is heavily influenced by Atkin and Lehner, and Atkin and Li's studies [1, 2] of twists of cuspidal newforms by Dirichlet characters, and their subsequent extensions to the Hilbert modular setting by Shemanske and Walling [14]. It should be noted that whereas the aforementioned papers examine the behavior of individual newforms under character twists, the behavior of the entire space generated by newforms under character twists has been studied by Hijikata, Pizer and Shemanske [6] in the elliptic modular setting and by the second author [10] in the Hilbert modular setting. We begin our study of twists of Hilbert Eisenstein series by proving the Eisenstein analogue of [2, Theorem 3.2]:

Proposition 1.3. *Let $f = E_{\psi_1,\psi_2}\in\mathscr{E}_k(\mathcal{N},\Psi)$ be a newform and Φ be a nontrivial Hecke character whose conductor is a power of a prime $\mathfrak{p}$. Then there exists a newform $g = E_{\psi_1\Phi^*,\psi_2\Phi^*}\in\mathscr{E}_k(\mathcal{L},\Psi\Phi^2)$ such that $f_\Phi = g - g\mid T_\mathfrak{p}\mid B_\mathfrak{p}$. Here f_Φ denotes the twist of f by Φ and $\mathcal{L} = \operatorname{cond}(\psi_1\Phi^*)\operatorname{cond}(\psi_2\Phi^*)$.*

As this proposition illustrates, the explicitness inherent to the study of Eisenstein series allows us to calculate the exact level of the twist of an Eisenstein newform by a Hecke character. This stands in sharp contrast with the cuspidal case, in which the exact level of the newform g cannot in general be identified. Using the above result we are able to show that if $f\in\mathscr{E}_k(\mathcal{N},\Psi)$ is a newform and Φ is a $\mathfrak{p}$-primary Hecke character where $(\mathfrak{p},\mathcal{N})=1$ then the twist of f by Φ is always a newform. In addition to extending a number of results of [14] from the cuspidal setting to the Eisenstein setting, we prove the following corollary, which does not in general hold for twists of cuspidal newforms.

Corollary 1.4. *Let $f\in\mathscr{E}_k(\mathcal{N},\Psi)$ be a newform with $\mathfrak{p}\nmid\operatorname{cond}(\Psi)$ and let Φ be a nontrivial $\mathfrak{p}$-primary Hecke character such that $2\cdot ord_\mathfrak{p}(\operatorname{cond}(\Phi))\neq ord_\mathfrak{p}(\mathcal{N})$. Then f_Φ is a newform.*

Our final theme, taken up in Section 6, is to use the newform theory developed in Section 3 in order to study Eisenstein series $f \in \mathscr{E}_k(\mathcal{N}, \Psi)$ which are eigenforms for some, but not all, of the Hecke operators $T_\mathfrak{p}$. Such Eisenstein series arise naturally in the theory of quadratic forms when one considers the genus theta series of a positive definite quadratic form (cf. [7]).

Let $S(f)$ be the set of finite primes of K for which f is not a $T_\mathfrak{p}$-eigenform. We call f the *obstruction set* of f. It is immediate that if f is a simultaneous eigenform for all of the Hecke operators $T_\mathfrak{p}$ then $S(f)$ is empty. We show in Proposition 6.4 that conversely, if the Dirichlet density $\delta(S(f))$ of $S(f)$ is less than or equal to $\frac{1}{2}$ then f is a simultaneous eigenform for all $T_\mathfrak{p}$ with $\mathfrak{p} \nmid \mathcal{N}$.

After defining the obstruction set of an Eisenstein series $f \in \mathscr{E}_k(\mathcal{N}, \Psi)$, we relate its density $\delta(S(f))$ to the number of classes of newforms of which f is a sum (where f is said to be a sum of t classes of newforms if f is a linear combination of shifts of t distinct newforms of level dividing $\mathcal{N}$). Let m be the order of the ideal class group of K modulo $\mathcal{N}\mathfrak{P}_\infty$. If $m = 1$ then there exists only a single class of newforms (as is the case, for instance, for Eisenstein series on the full modular group $SL_2(\mathbb{Z})$). We therefore suppose that $m > 1$ and prove the following theorem (Theorem 6.6):

Theorem 1.5. *Suppose that $\delta(S(f)) = \frac{x}{m}$ for some integer $0 \leq x < m$. Then f is a linear combination of at most $\lfloor \frac{m}{m-x} \rfloor$ classes of newforms.*

It is our pleasure to thank Tom Shemanske for his encouragement and for commenting on several drafts of this paper and Michael Wijaya for several helpful conversations regarding the material contained in Section 6.

2. Notation

For the most part we follow the notation of [14, 15, 16]. However, to make this paper somewhat self-contained, we shall briefly review the basic definitions of the functions and operators which we shall study.

Let K be a totally real number field of degree n over $\mathbb{Q}$ with ring of integers $\mathcal{O}$, group of units $\mathcal{O}^\times$, and totally positive units $\mathcal{O}_+^\times$. Let $\mathfrak{d}$ be the different of K. If $\mathfrak{q}$ is a finite prime of K, we denote by $K_\mathfrak{q}$ the completion of K at $\mathfrak{q}$, $\mathcal{O}_\mathfrak{q}$ the valuation ring of $K_\mathfrak{q}$, and $\pi_\mathfrak{q}$ a local uniformizer. For a set S of finite primes of K, we denote by $\delta(S)$ the Dirichlet density of S relative to the set of all finite primes of K. Finally, we let $\mathfrak{P}_\infty$ denote the product of all archimedean primes of K.

We denote by $K_\mathbb{A}$ the ring of K-adeles and by $K_\mathbb{A}^\times$ the group of K-ideles. As usual we view K as a subgroup of $K_\mathbb{A}$ via the diagonal embedding. If $\tilde{\alpha} \in K_\mathbb{A}^\times$, we let $\tilde{\alpha}_\infty$ denote the archimedean part of $\tilde{\alpha}$ and $\tilde{\alpha}_0$ the finite part of $\tilde{\alpha}$. If $\mathcal{J}$ is an integral ideal we let $\tilde{\alpha}_\mathcal{J}$ denote the $\mathcal{J}$-part of $\tilde{\alpha}$.

For an integral ideal $\mathcal{N}$ we define a numerical character ϕ modulo $\mathcal{N}$ to be a homomorphism $\phi : (\mathcal{O}/\mathcal{N})^\times \to \mathbb{C}^\times$, and a Hecke character to be a continuous character on the idele class group: $\Phi : K_{\mathbb{A}}^\times / K^\times \to \mathbb{C}^\times$. We denote the induced character on $K_{\mathbb{A}}^\times$ by Φ as well. Every Hecke character is of the form $\Phi(\tilde{\alpha}) = \prod_\nu \Phi_\nu(\alpha_\nu)$ where Φ_ν is a character $\Phi_\nu : K_\nu^\times \longrightarrow \mathbb{C}^\times$. The conductor, cond$(\Phi)$, of Φ is defined to be the modulus whose finite part is $\mathfrak{f}_\Phi$ (see [5]) and whose infinite part is the formal product of those archimedean primes ν for which Φ_ν is nontrivial. We adopt the convention that ϕ and ψ will always denote numerical characters and Φ and Ψ will denote Hecke characters.

For a fractional ideal $\mathcal{I}$ and integral ideal $\mathcal{N}$, define

$$\Gamma_0(\mathcal{N}, \mathcal{I}) = \left\{ A \in \begin{pmatrix} \mathcal{O} & \mathcal{I}^{-1}\mathfrak{d}^{-1} \\ \mathcal{N}\mathcal{I}\mathfrak{d} & \mathcal{O} \end{pmatrix} : \det A \in \mathcal{O}_+^\times \right\}.$$

Let $\theta : \mathcal{O}_+^\times \to \mathbb{C}^\times$ be a character of finite order and note that there exists an element $m \in \mathbb{R}^n$ such that $\theta(a) = a^{im}$ for all totally positive units a. While such an m is not unique, we shall fix one such m for the remainder of this paper.

Let $k = (k_1, ..., k_n) \in \mathbb{Z}_+^n$ and ϕ be a numerical character modulo $\mathcal{N}$. Following Shimura [15, 16], we define $M_k(\Gamma_0(\mathcal{N}, \mathcal{I}), \phi, \theta)$ to be the complex vector space of classical Hilbert modular forms on $\Gamma_0(\mathcal{N}, \mathcal{I})$.

It is well-known that the space classical Hilbert modular forms of a fixed weight, character and congruence subgroup is not invariant under the algebra generated by the Hecke operators $T_\mathfrak{p}$. We therefore consider the larger space of adelic Hilbert modular forms, which *is* invariant under the Hecke algebra. Our construction follows that of Shimura [16].

Fix a set of strict ideal class representatives $\mathcal{I}_1, ..., \mathcal{I}_h$ of K, set $\Gamma_\lambda = \Gamma_0(\mathcal{N}, \mathcal{I}_\lambda)$, and put

$$\mathfrak{M}_k(\mathcal{N}, \phi, \theta) = \prod_{\lambda=1}^{h} M_k(\Gamma_\lambda, \phi, \theta).$$

Let $G_{\mathbb{A}} = GL_2(K_{\mathbb{A}})$ and view $G_K = GL_2(K)$ as a subgroup of $G_{\mathbb{A}}$ via the diagonal embedding. Denote by $G_\infty = GL_2(\mathbb{R})^n$ the archimedean part of $G_{\mathbb{A}}$. For an integral ideal $\mathcal{N}$ of $\mathcal{O}$ and a prime $\mathfrak{p}$, let

$$Y_\mathfrak{p}(\mathcal{N}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathcal{O}_\mathfrak{p} & \mathfrak{d}^{-1}\mathcal{O}_\mathfrak{p} \\ \mathcal{N}\mathfrak{d}\mathcal{O}_\mathfrak{p} & \mathcal{O}_\mathfrak{p} \end{pmatrix} : \det A \in K_\mathfrak{p}^\times, (a\mathcal{O}_\mathfrak{p}, \mathcal{N}\mathcal{O}_\mathfrak{p}) = 1 \right\},$$

$$W_\mathfrak{p}(\mathcal{N}) = \{ A \in Y_\mathfrak{p}(\mathcal{N}) : \det A \in \mathcal{O}_\mathfrak{p}^\times \}$$

and put

$$Y = Y(\mathcal{N}) = G_{\mathbb{A}} \cap \left(G_{\infty+} \times \prod_{\mathfrak{p} \nmid \mathfrak{P}_\infty} Y_\mathfrak{p}(\mathcal{N}) \right), \qquad W = W(\mathcal{N}) = G_{\infty+} \times \prod_{\mathfrak{p} \nmid \mathfrak{P}_\infty} W_\mathfrak{p}(\mathcal{N}).$$

Given a numerical character ϕ modulo $\mathcal{N}$ define a homomorphism $\phi_Y : Y \to \mathbb{C}^\times$ by setting $\phi_Y(\begin{pmatrix} \tilde{a} & * \\ * & * \end{pmatrix}) = \phi(\tilde{a}_{\mathcal{N}} \bmod \mathcal{N})$.

Given a fractional ideal $\mathcal{I}$ of K define $\tilde{\mathcal{I}} = (\mathcal{I}_\nu)_\nu$ to be a fixed idele such that $\mathcal{I}_\infty = 1$ and $\tilde{\mathcal{I}}\mathcal{O} = \mathcal{I}$. For $\lambda = 1, ..., h$, set $x_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{I}_\lambda \end{pmatrix} \in G_{\mathbb{A}}$. By the Strong Approximation theorem

$$G_{\mathbb{A}} = \bigcup_{\lambda=1}^{h} G_K x_\lambda W = \bigcup_{\lambda=1}^{h} G_K x_\lambda^{-\iota} W,$$

where ι denotes the canonical involution on two-by-two matrices.

For an h-tuple $(f_1, ..., f_h) \in \mathfrak{M}_k(\mathcal{N}, \phi, \theta)$ we define a function $\mathbf{f} : G_{\mathbb{A}} \to \mathbb{C}$ by

$$\mathbf{f}(\alpha x_\lambda^{-\iota} w) = \phi_Y(w^\iota) \det(w_\infty)^{im} (f_\lambda \mid w_\infty)(\mathbf{i})$$

for $\alpha \in G_K$, $w \in W(\mathcal{N})$ and $\mathbf{i} = (i, ..., i)$ (with $i = \sqrt{-1}$). Here

$$f_\lambda \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau) = (ad - bc)^{\frac{k}{2}} (c\tau + d)^{-k} f_\lambda \left(\frac{a\tau + b}{c\tau + d} \right).$$

We identify $\mathfrak{M}_k(\mathcal{N}, \phi, \theta)$ with the set of functions $\mathbf{f} : G_{\mathbb{A}} \to \mathbb{C}$ satisfying

(1) $\mathbf{f}(\alpha x w) = \phi_Y(w^\iota)\, \mathbf{f}(x)$ for all $\alpha \in G_K, x \in G_{\mathbb{A}}, w \in W(\mathcal{N}), w_\infty = 1$
(2) For each λ there exists an element $f_\lambda \in M_k$ such that

$$\mathbf{f}(x_\lambda^{-\iota} y) = \det(y)^{im} (f_\lambda \mid y)(\mathbf{i})$$

for all $y \in G_{\infty+}$.

Let $\phi_\infty : K_{\mathbb{A}}^\times \to \mathbb{C}^\times$ be defined by $\phi_\infty(\tilde{a}) = \mathrm{sgn}(\tilde{a}_\infty)^k |\tilde{a}_\infty|^{2im}$, where m was specified in the definition of θ. We say that a Hecke character Φ extends $\phi\phi_\infty$ if $\Phi(\tilde{a}) = \phi(\tilde{a}_{\mathcal{N}} \bmod \mathcal{N})\phi_\infty(\tilde{a})$ for all $\tilde{a} \in K_\infty^\times \times \prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^\times$. If the previous equality holds for $\phi_\infty(\tilde{a}) = \mathrm{sgn}(\tilde{a}_\infty)^k$ then we say that Φ extends ϕ. Any Hecke character Φ extending $\phi\phi_\infty$ has conductor dividing $\mathcal{N}\mathfrak{P}_\infty$. Henceforth we will use the word conductor to refer to the finite part of the conductor.

Given a Hecke character Φ extending $\phi\phi_\infty$ we define an ideal character Φ^* modulo $\mathcal{N}\mathfrak{P}_\infty$ by

$$\begin{cases} \Phi^*(\mathfrak{p}) = \Phi(\tilde{\pi}_{\mathfrak{p}}) & \text{for } \mathfrak{p} \nmid \mathcal{N} \text{ and } \tilde{\pi}\mathcal{O} = \mathfrak{p}, \\ \Phi^*(\mathfrak{a}) = 0 & \text{if } (\mathfrak{a}, \mathcal{N}) \neq 1 \end{cases}$$

For $\tilde{s} \in K_{\mathbb{A}}^\times$, define $\mathbf{f}^{\tilde{s}}(x) = \mathbf{f}(\tilde{s}x)$. The map $\tilde{s} \longrightarrow \left(\mathbf{f} \mapsto \mathbf{f}^{\tilde{s}}\right)$ defines a unitary representation of $K_{\mathbb{A}}^\times$ in $\mathfrak{M}_k(\mathcal{N}, \phi, \theta)$. By Schur's Lemma the irreducible subrepresentations are all one-dimensional (since $K_{\mathbb{A}}^\times$ is abelian). For a character Φ on $K_{\mathbb{A}}^\times$, let $\mathscr{M}_k(\mathcal{N}, \Phi)$ denote the subspace of $\mathfrak{M}_k(\mathcal{N}, \phi, \theta)$ consisting of all $\mathbf{f}$ for which $\mathbf{f}^{\tilde{s}} = \Phi(\tilde{s})\, \mathbf{f}$ and let $\mathscr{S}_k(\mathcal{N}, \Phi) \subset \mathscr{M}_k(\mathcal{N}, \Phi)$ denote the subspace of cusp forms. If $s \in K^\times$ then $\mathbf{f}^s = \mathbf{f}$. It follows that $\mathscr{M}_k(\mathcal{N}, \Phi)$ is nonempty only when Φ is a Hecke character.

If $\mathbf{f} = (f_1, ..., f_h) \in \mathfrak{M}_k(\mathcal{N}, \phi, \theta)$, then each f_λ has a Fourier expansion

$$f_\lambda(\tau) = a_\lambda(0) + \sum_{0 \ll \xi \in \mathcal{I}_\lambda} a_\lambda(\xi) e^{2\pi i \mathrm{tr}(\xi\tau)}.$$

If $\mathfrak{m}$ is an integral ideal then we define the $\mathfrak{m}$-th 'Fourier' coefficient of $\mathbf{f}$ by

$$C(\mathfrak{m}, \mathbf{f}) = \begin{cases} N(\mathfrak{m})^{\frac{k_0}{2}} a_\lambda(\xi) \xi^{-\frac{k}{2} - im} & \text{if } \mathfrak{m} = \xi \mathcal{I}_\lambda^{-1} \subset \mathcal{O} \\ 0 & \text{otherwise} \end{cases}$$

where $k_0 = \max\{k_1, ..., k_n\}$.

Given $\mathbf{f} \in \mathfrak{M}_k(\mathcal{N}, \phi, \theta)$ and $y \in G_{\mathbb{A}}$ define a slash operator by setting $(\mathbf{f} \mid y)(x) = \mathbf{f}(xy^\iota)$.

For an integral ideal $\mathfrak{r}$ define the shift operator $B_\mathfrak{r}$ by

$$\mathbf{f} \mid B_\mathfrak{r} = N(\mathfrak{r})^{-\frac{k_0}{2}} \mathbf{f} \mid \begin{pmatrix} 1 & 0 \\ 0 & \tilde{\mathfrak{r}}^{-1} \end{pmatrix}.$$

The shift operator maps $\mathscr{M}_k(\mathcal{N}, \Phi)$ to $\mathscr{M}_k(\mathfrak{r}\mathcal{N}, \Phi)$. Further, $C(\mathfrak{m}, \mathbf{f} \mid B_\mathfrak{r}) = C(\mathfrak{m}\mathfrak{r}^{-1}, \mathbf{f})$. It is clear that $\mathbf{f} \mid B_{\mathfrak{r}_1} \mid B_{\mathfrak{r}_2} = \mathbf{f} \mid B_{\mathfrak{r}_1\mathfrak{r}_2}$.

For an integral ideal $\mathfrak{r}$ the Hecke operator $T_\mathfrak{r} = T_\mathfrak{r}^\mathcal{N}$ maps $\mathscr{M}_k(\mathcal{N}, \Phi)$ to itself regardless of whether or not $(\mathfrak{r}, \mathcal{N}) = 1$. This action is given on Fourier coefficients by

$$C(\mathfrak{m}, \mathbf{f} \mid T_\mathfrak{r}) = \sum_{\mathfrak{m} + \mathfrak{r} \subset \mathfrak{a}} \Phi^*(\mathfrak{a}) N(\mathfrak{a})^{k_0 - 1} C(\mathfrak{a}^{-2}\mathfrak{m}\mathfrak{r}, \mathbf{f}).$$

Note that if $(\mathfrak{a}, \mathfrak{r}) = 1$ then $B_\mathfrak{a} T_\mathfrak{r} = T_\mathfrak{r} B_\mathfrak{a}$.

3. Hilbert Eisenstein newforms

Fix a space $\mathscr{M}_k(\mathcal{N}, \Psi) \subset \mathfrak{M}_k(\mathcal{N}, \psi)$ where $\psi : (\mathcal{O}_K/\mathcal{N})^\times \to \mathbb{C}^\times$ is a numerical character, Ψ is a Hecke character extending ψ and $k \in \mathbb{Z}^n$. Let $\mathscr{E}_k(\mathcal{N}, \Psi)$ be the orthogonal complement of $\mathscr{S}_k(\mathcal{N}, \Psi)$ in $\mathscr{M}_k(\mathcal{N}, \Psi)$ with respect to the Petersson inner product of [16, (2.27)] (i.e. $\mathscr{E}_k(\mathcal{N}, \Psi)$ is the subspace of Eisenstein series). It is well known that $\mathscr{M}_k(\mathcal{N}, \Psi) = \mathscr{S}_k(\mathcal{N}, \Psi)$ unless $k_1 = \cdots = k_n \geq 0$. Thus we may abuse notation and make the identification $k = (k, ..., k)$ for some integer k. We will assume throughout that $k \geq 3$. In this section we will develop a newform theory for $\mathscr{E}_k(\mathcal{N}, \Psi)$ which parallels the newform theory of $\mathscr{S}_k(\mathcal{N}, \Psi)$.

We begin by constructing the forms which will be our main object of study.

Proposition 3.1. *Let $\mathcal{N}_1, \mathcal{N}_2$ be integral ideals and ψ_1 (respectively ψ_2) be a character, not necessarily primitive, on the ideal class group modulo $\mathcal{N}_1\mathfrak{P}_\infty$ (respectively $\mathcal{N}_2\mathfrak{P}_\infty$) such that*

$$\psi_1(\nu\mathcal{O}) = sgn(\nu)^a \qquad \text{for } \nu \equiv 1 \pmod{\mathcal{N}_1}$$
$$\psi_2(\nu\mathcal{O}) = sgn(\nu)^b \qquad \text{for } \nu \equiv 1 \pmod{\mathcal{N}_2},$$

where $a, b \in \mathbb{Z}^n$ *and* $a+b \equiv k \pmod{2\mathbb{Z}^n}$*. Then there exists* $E_{\psi_1,\psi_2} \in \mathscr{M}_k(\mathcal{N}_1\mathcal{N}_2, \Psi)$*, where* Ψ *is the Hecke character such that* $\Psi^*(\mathfrak{r}) = (\psi_1\psi_2)(\mathfrak{r})$ *for* $\mathfrak{r}$ *coprime to* $\mathcal{N}_1\mathcal{N}_2$*, such that for any integral ideal* $\mathfrak{m}$*,*

$$C(\mathfrak{m}, E_{\psi_1,\psi_2}) = \sum_{\mathfrak{r}|\mathfrak{m}} \psi_1(\mathfrak{m}\mathfrak{r}^{-1})\psi_2(\mathfrak{r})N(\mathfrak{r})^{k-1}. \tag{1}$$

Proof. This is Proposition 3.4 of [16]. □

Remark 3.2. The forms E_{ψ_1,ψ_2} of Proposition 3.1 were studied in Section 2.2 of [4], where explicit formulae for their constant terms were derived.

The following proposition is easily verified by examining 'Fourier coefficients'.

Proposition 3.3. *Let* E_{ψ_1,ψ_2} *be as above. Then*

$$E_{\psi_1,\psi_2} \mid T_{\mathfrak{p}} = C(\mathfrak{p}, E_{\psi_1,\psi_2})E_{\psi_1,\psi_2}$$

for all primes $\mathfrak{p}$ *not dividing* $\mathcal{N}_1\mathcal{N}_2$*.*

Let $S(\mathcal{N}, \Psi)$ be the set consisting of all triples $(\psi_1, \psi_2, \mathfrak{a})$ where ψ_1 (respectively ψ_2) is a character mod $\mathcal{N}_1\mathfrak{P}_\infty$ (respectively mod $\mathcal{N}_2\mathfrak{P}_\infty$) for integral ideals $\mathcal{N}_1, \mathcal{N}_2$ satisfying $\mathcal{N}_1\mathcal{N}_2 \mid \mathcal{N}$ and $\Psi^*(\mathfrak{r}) = (\psi_1\psi_2)(\mathfrak{r})$ for $\mathfrak{r}$ coprime to $\mathcal{N}_1\mathcal{N}_2$, and where $\mathfrak{a}$ divides $\mathcal{N}\mathcal{N}_1^{-1}\mathcal{N}_2^{-1}$. Let $E_k(\mathcal{N}, \Psi) = \{E_{\psi_1,\psi_2} \mid B_{\mathfrak{a}} : (\psi_1, \psi_2, \mathfrak{a}) \in S(\mathcal{N}, \Psi)\}$.

We are now in a position to define our principal object of study.

Definition 3.4. As in [17, 18] we say that a form E_{ψ_1,ψ_2} is a newform of level $\mathcal{N}$ if the characters ψ_1, ψ_2 are primitive and $\mathrm{cond}(\psi_1)\mathrm{cond}(\psi_2) = \mathcal{N}$. We will denote by $\mathscr{E}_k^{(new)}(\mathcal{N}, \Psi)$ the subspace of $\mathscr{M}_k(\mathcal{N}, \Psi)$ generated by the newforms of level $\mathcal{N}$.

Remark 3.5. The notation $\mathscr{E}_k^{(new)}(\mathcal{N}, \Psi)$ may seem confusing because it seems to imply that $E_k(\mathcal{N}, \Psi) \subset \mathscr{E}_k(\mathcal{N}, \Psi)$, an assertion which we have not yet proven. This will be proven as Proposition 3.9.

As in the cuspidal case, the eigenvalues of a newform with respect to $\{T_{\mathfrak{p}} : \mathfrak{p} \nmid \mathcal{N}\}$ distinguish it from any other newform. We make this precise in the following theorem.

Theorem 3.6. *Let* $\mathcal{M}, \mathcal{N}$ *be integral ideals and* $E_{\psi_1,\psi_2} \in \mathscr{E}_k^{(new)}(\mathcal{M}, \Psi)$ *and* $E_{\phi_1,\phi_2} \in \mathscr{E}_k^{(new)}(\mathcal{N}, \Psi)$ *be newforms such that*

$$C(\mathfrak{p}, E_{\psi_1,\psi_2}) = C(\mathfrak{p}, E_{\phi_1,\phi_2})$$

for a set of primes having Dirichlet density strictly greater than $\frac{1}{2}$*. Then* $\mathcal{M} = \mathcal{N}$ *and* $E_{\psi_1,\psi_2} = E_{\phi_1,\phi_2}$*.*

In order to prove our theorem we will use the following lemma.

Lemma 3.7. *Let $\chi_1, \chi_2, \chi_3, \chi_4$ be characters on the ideal class group modulo $\mathcal{MN}\mathfrak{P}_\infty$ such that $\chi_1\chi_2 = \chi_3\chi_4$. Then there exists a constant $\kappa > 0$ such that if $\mathfrak{p}$ is a prime not dividing $\mathcal{MN}$ with $N(\mathfrak{p}) > \kappa$ and*

$$\chi_1(\mathfrak{p}) + \chi_2(\mathfrak{p})N(\mathfrak{p})^{k-1} = \chi_3(\mathfrak{p}) + \chi_4(\mathfrak{p})N(\mathfrak{p})^{k-1},$$

then $\chi_1(\mathfrak{p}) = \chi_3(\mathfrak{p})$ and $\chi_2(\mathfrak{p}) = \chi_4(\mathfrak{p})$.

Proof. Suppose that $\mathfrak{p}$ is a prime not dividing $\mathcal{MN}$ for which $\chi_1(\mathfrak{p}) + \chi_2(\mathfrak{p})N(\mathfrak{p})^{k-1} = \chi_3(\mathfrak{p}) + \chi_4(\mathfrak{p})N(\mathfrak{p})^{k-1}$. Then $\chi_1(\mathfrak{p}) - \chi_3(\mathfrak{p}) = (\chi_4(\mathfrak{p}) - \chi_2(\mathfrak{p}))\, N(\mathfrak{p})^{k-1}$. If $\chi_2(\mathfrak{p}) = \chi_4(\mathfrak{p})$ then our hypothesis that $\chi_1\chi_2 = \chi_3\chi_4$ shows that $\chi_1(\mathfrak{p}) = \chi_3(\mathfrak{p})$ as well and we have nothing to show. We therefore suppose that $\chi_2(\mathfrak{p}) \neq \chi_4(\mathfrak{p})$. In this case $N(\mathfrak{p})^{k-1} = \frac{\chi_1(\mathfrak{p})-\chi_3(\mathfrak{p})}{\chi_4(\mathfrak{p})-\chi_2(\mathfrak{p})}$. Our lemma now follows from the observation that the absolute value of the latter fraction can be bounded independently of $\mathfrak{p}$ whereas $N(\mathfrak{p})^{k-1}$ cannot. □

We now give a proof of Theorem 3.6

Proof. Let $\mathfrak{p}_1, ..., \mathfrak{p}_m$ represent the ideal classes modulo $\mathcal{MN}\mathfrak{P}_\infty$. By the Chebotarev Density theorem we may assume that for each i we have $N(\mathfrak{p}_i) > \kappa$ where κ is the constant appearing in Lemma 3.7. Denote by $\psi_1', \psi_2', \phi_1', \phi_2'$ the characters on the ideal class group modulo $\mathcal{MN}\mathfrak{P}_\infty$ induced by $\psi_1, \psi_2, \phi_1, \phi_2$.

Suppose that $C(\mathfrak{p}_i, E_{\psi_1,\psi_2}) = C(\mathfrak{p}_i, E_{\phi_1,\phi_2})$ so that $\psi_1(\mathfrak{p}_i) + \psi_2(\mathfrak{p}_i)N(\mathfrak{p}_i)^{k-1} = \phi_1(\mathfrak{p}_i) + \phi_2(\mathfrak{p}_i)N(\mathfrak{p}_i)^{k-1}$. The same equality must hold with the induced characters $\psi_1', \psi_2', \phi_1', \phi_2'$, so it follows from Lemma 3.7 that $\psi_1'(\mathfrak{p}_i) = \phi_1'(\mathfrak{p}_i)$ and $\psi_2'(\mathfrak{p}_i) = \phi_2'(\mathfrak{p}_i)$. If $\psi_1'(\mathfrak{p}_i) = \phi_1'(\mathfrak{p}_i)$ for s values of i then the density of primes for which $C(\mathfrak{p}, E_{\psi_1,\psi_2}) = C(\mathfrak{p}, E_{\phi_1,\phi_2})$ is at most $\frac{s}{m}$. It now follows that $\psi_1'(\mathfrak{p}_i) = \phi_1'(\mathfrak{p}_i)$ for more than $\frac{m}{2}$ values of i. Consider the character $\chi = \psi_1'\overline{\phi_1'}$. The orthogonality relations on characters show that $\sum_{i=1}^m \chi(\mathfrak{p}_i) = 0$ unless χ is the principal character. We have shown that χ assumes the value 1 for more than $\frac{m}{2}$ of the $\mathfrak{p}_i$, hence χ is principal and $\psi_1' = \phi_1'$. As $\psi_1'\psi_2' = \phi_1'\phi_2'$ we see that $\psi_2' = \phi_2'$ as well. Because E_{ψ_1,ψ_2} and E_{ϕ_1,ϕ_2} are newforms, $\psi_1, \psi_2, \phi_1, \phi_2$ are all primitive, hence $\psi_1 = \phi_1$ and $\psi_2 = \phi_2$. This shows that $E_{\psi_1,\psi_2} = E_{\phi_1,\phi_2}$. That $\mathcal{M} = \mathcal{N}$ now follows from the equalities $\mathcal{M} = \mathrm{cond}(\psi_1)\mathrm{cond}(\psi_2) = \mathrm{cond}(\phi_1)\mathrm{cond}(\phi_2) = \mathcal{N}$. □

Having proven Theorem 3.6, we are in a position to prove

Proposition 3.8. *The elements of $E_k(\mathcal{N}, \Psi)$ are linearly independent over $\mathbb{C}$.*

Proof. Let $f_1, \ldots, f_r \in E_k(\mathcal{N}, \Psi)$ be distinct and such that

(2) $$\sum_{i=1}^{r} c_i f_i = 0$$

where the c_i are nonzero complex numbers and r is minimal in the sense that any subset of $E_k(\mathcal{N},\Psi)$ with fewer than r elements is linearly independent (over $\mathbb{C}$).

Let $\mathfrak{p}$ be a prime not dividing $\mathcal{N}$ and note that all of the f_i are eigenforms for $T_\mathfrak{p}$. Moreover, each f_i is of the form $F^{(i)}_{\psi_1^{(i)},\psi_2^{(i)}} \mid B_{\mathfrak{a}_i}$ for some newform $F^{(i)}_{\psi_1^{(i)},\psi_2^{(i)}} \in E_k(\mathrm{cond}(\psi_1^{(i)})\mathrm{cond}(\psi_2^{(i)}),\psi)$. As the operators $B_{\mathfrak{a}_i}$ and $T_\mathfrak{p}$ commute whenever $\mathfrak{p}\nmid\mathfrak{a}_i$, the $T_\mathfrak{p}$-eigenvalue of f_i is $C(\mathfrak{p},F^{(i)}_{\psi_1^{(i)},\psi_2^{(i)}})$.

Apply the operator $\left(T_\mathfrak{p} - C(\mathfrak{p},F^{(1)}_{\psi_1^{(1)},\psi_2^{(1)}})Id\right)$ to equation (2) to get

$$\sum_{i=2}^{r} c_i\left(C(\mathfrak{p},F^{(i)}_{\psi_1^{(i)},\psi_2^{(i)}}) - C(\mathfrak{p},F^{(1)}_{\psi_1^{(1)},\psi_2^{(1)}})\right) f_i = 0. \tag{3}$$

The minimality of r implies that all of the coefficients in equation (3) are zero; that is,

$$C(\mathfrak{p},F^{(i)}_{\psi_1^{(i)},\psi_2^{(i)}}) = C(\mathfrak{p},F^{(1)}_{\psi_1^{(1)},\psi_2^{(1)}}) \qquad \text{for } i=1,\dots,r.$$

Theorem 3.6 shows that all of the $F^{(i)}_{\psi_1^{(i)},\psi_2^{(i)}}$ are equal. Therefore equation (2) provides a dependence relation amongst various shifts of a single newform. This contradicts [14, Proposition 3.2] (whose proof applies in this context) which implies that these elements are linearly independent. □

We continue our analysis of $E_k(\mathcal{N},\Psi)$ by showing that $E_k(\mathcal{N},\Psi)\subset\mathscr{E}_k(\mathcal{N},\Psi)$.

Proposition 3.9. *The subspace of $\mathscr{M}_k(\mathcal{N},\Psi)$ generated by $E_k(\mathcal{N},\Psi)$ is orthogonal to $\mathscr{S}_k(\mathcal{N},\Psi)$ with respect to the Petersson inner product and thus consists of Eisenstein series.*

Proof. The newform theory of $\mathscr{S}_k(\mathcal{N},\Psi)$ (see [14]) shows that it suffices to prove, for every $E_{\psi_1,\psi_2}\mid B_\mathfrak{a}\in E_k(\mathcal{N},\Psi)$, that $E_{\psi_1,\psi_2}\mid B_\mathfrak{a}$ is orthogonal to every element of the form $G\mid B_\mathfrak{b}$ with $G\in\mathscr{S}_k(\mathcal{M},\Psi)$ a cuspidal newform, $\mathcal{M}\mid\mathcal{N}$, $\mathfrak{a}\mid\mathcal{N}$ and $\mathfrak{b}\mid\mathcal{N}\mathcal{M}^{-1}$. Observe that by Proposition 2.4 of [16] we have, for $\mathfrak{p}\nmid\mathcal{N}$,

$$\langle E_{\psi_1,\psi_2}\mid B_\mathfrak{a}\mid T_\mathfrak{p}, G\mid B_\mathfrak{b}\rangle = \Psi^*(\mathfrak{p})\langle E_{\psi_1,\psi_2}\mid B_\mathfrak{a}, G\mid B_\mathfrak{b}\mid T_\mathfrak{p}\rangle.$$

It follows that

$$|C(\mathfrak{p},E_{\psi_1,\psi_2})\cdot\langle E_{\psi_1,\psi_2}\mid B_\mathfrak{a}, G\mid B_\mathfrak{b}\rangle| = |C(\mathfrak{p},G)\cdot\langle E_{\psi_1,\psi_2}\mid B_\mathfrak{a}, G\mid B_\mathfrak{b}\rangle|.$$

If $\langle E_{\psi_1,\psi_2}\mid B_\mathfrak{a}, G\mid B_\mathfrak{b}\rangle = 0$ then we are done, so we assume to the contrary that $\langle E_{\psi_1,\psi_2}\mid B_\mathfrak{a}, G\mid B_\mathfrak{b}\rangle \neq 0$. Therefore $|C(\mathfrak{p},E_{\Psi_1,\Psi_2})| = |C(\mathfrak{p},G)|$. We use this equality to arrive at a contradiction.

On the one hand, Shahidi [13] has shown that $|C(\mathfrak{p},G)|\leq 2N(\mathfrak{p})^{\frac{k-1}{2}+\frac{1}{5}}$.

On the other hand, it is clear that for $\mathfrak{p} \nmid \mathcal{N}$ we have

$$|C(\mathfrak{p}, E_{\psi_1,\psi_2})| = |\psi_1(\mathfrak{p}) + \psi_2(\mathfrak{p})N(\mathfrak{p})^{k-1}| \geq N(\mathfrak{p})^{k-1} - 1.$$

This contradiction finishes our proof. □

We have just shown that the subspace of $\mathscr{M}_k(\mathcal{N}, \Psi)$ generated by $E_k(\mathcal{N}, \Psi)$ is a subspace of $\mathscr{E}_k(\mathcal{N}, \Psi)$. In fact, a stronger statement is true.

Proposition 3.10. *The subspace of $\mathscr{M}_k(\mathcal{N}, \Psi)$ generated by $E_k(\mathcal{N}, \Psi)$ is equal to $\mathscr{E}_k(\mathcal{N}, \Psi)$.*

Proof. Proposition 3.9 shows that the subspace of $\mathscr{M}_k(\mathcal{N}, \Psi)$ generated by $E_k(\mathcal{N}, \Psi)$ lies in $\mathscr{E}_k(\mathcal{N}, \Psi)$, so it suffices to show that these spaces have equal dimensions. In the case $k = 2$ this was proven by Wiles [18, Proposition 1.5]. Wiles notes that a similar proof holds for (parallel) weights $k > 2$. □

We have seen that the subspace of $\mathscr{E}_k(\mathcal{N}, \Psi)$ generated by $E_k(\mathcal{N}, \Psi)$ is precisely $\mathscr{E}_k(\mathcal{N}, \Psi)$, the subspace of Eisenstein series. Our discussion thus far makes clear the following decomposition of $\mathscr{E}_k(\mathcal{N}, \Psi)$.

Proposition 3.11. *Notation as above, we have the following decomposition of $\mathscr{E}_k(\mathcal{N}, \Psi)$:*

$$\mathscr{E}_k(\mathcal{N}, \Psi) = \bigoplus_{\mathfrak{f}_\Psi | \mathcal{M} | \mathcal{N}} \quad \bigoplus_{\mathfrak{r} | \mathcal{N}\mathcal{M}^{-1}} \mathscr{E}_k^{(new)}(\mathcal{M}, \Psi) \mid B_{\mathfrak{r}}.$$

Proof. This follows from Propositions 3.8 and 3.9. □

Remark 3.12. Proposition 3.11 shows that every element of $\mathscr{E}_k(\mathcal{N}, \Psi)$ can be written uniquely as a linear combination of shifts of newforms (of possibly lower level). Fix a prime $\mathfrak{p}$ which represents the trivial ideal class modulo $\mathcal{N}\mathfrak{P}_\infty$ and an integral ideal $\mathcal{M}$ which divides $\mathcal{N}$. By Proposition 3.3 every newform $E_{\psi_1,\psi_2} \in \mathscr{E}_k(\mathcal{M}, \Psi)$ is a $T_\mathfrak{p}$-eigenform with eigenvalue $C(\mathfrak{p}, E_{\psi_1,\psi_2}) = \psi_1(\mathfrak{p}) + \psi_2(\mathfrak{p})N(\mathfrak{p})^{k-1} = 1 + N(\mathfrak{p})^{k-1}$. As $\mathfrak{p}$ is coprime to $\mathcal{N}$, $T_\mathfrak{p}$ commutes with the shift operator $B_\mathfrak{r}$ for every ideal $\mathfrak{r} \mid \mathcal{N}\mathcal{M}^{-1}$. From this we conclude that every element of $\mathscr{E}_k(\mathcal{N}, \Psi)$ is a Hecke eigenform for $T_\mathfrak{p}$. In particular, every element of $\mathscr{E}_k(\mathcal{N}, \Psi)$ is a Hecke eigenform for a set of primes of K with positive density.

As in the cuspidal case, we shall say that two elements $E, F \in \mathscr{E}_k(\mathcal{N}, \Psi)$ which are both simultaneous eigenforms for all Hecke operators $T_\mathfrak{p}$ with $\mathfrak{p} \nmid \mathcal{N}$ are equivalent, to be denoted $E \sim F$, if they have equal $T_\mathfrak{p}$-eigenvalues for all primes $\mathfrak{p}$ with $\mathfrak{p} \nmid \mathcal{N}$. We now show that every Hecke eigenform $F \in \mathscr{E}_k(\mathcal{N}, \Psi)$ is equivalent to a newform of level $\mathcal{M}$ for some ideal $\mathcal{M} \mid \mathcal{N}$.

Proposition 3.13. *Let $F \in \mathscr{E}_k(\mathcal{N}, \Psi)$ be a nonzero Hecke eigenform for all $T_\mathfrak{p}$ with $\mathfrak{p} \nmid \mathcal{N}$. Then*

1. *(1) There exists an ideal $\mathcal{M} \mid \mathcal{N}$ and a unique newform G of level $\mathcal{M}$ such that $F \sim G$.*
2. *(2) F lies in the vector space spanned by $\{G \mid B_\mathfrak{a} : \mathfrak{a} \mid \mathcal{N}\mathcal{M}^{-1}\}$.*

Proof. If $\mathfrak{p} \nmid \mathcal{N}$ then denote by $\lambda_\mathfrak{p}$ the $T_\mathfrak{p}$-eigenvalue of F.

Let $\{G_i\}$ be the set of all newforms of level dividing $\mathcal{N}$. For each i, let $\mathcal{L}_i$ be the level of G_i. As the proposition is obvious if $\#\{G_i\} = 1$, we assume that $\{G_i\}$ contains at least two elements. Suppose now that for each G_i there exists a prime $\mathfrak{p}_i \nmid \mathcal{N}$ such that $C(\mathfrak{p}_i, G_i) \neq \lambda_{\mathfrak{p}_i}$. Write

$$F = \sum_i \sum_{\mathfrak{a}_i \mid \mathcal{N}\mathcal{L}_i^{-1}} c_i (G_i \mid B_{\mathfrak{a}_i})$$

where each $c_i \in \mathbb{C}$ is a constant. Such a decomposition is possible because $E_k(\mathcal{N}, \Psi)$ is a basis for $\mathscr{E}_k(\mathcal{N}, \Psi)$. Define the operator $T = \prod_i (T_{\mathfrak{p}_i} - C(\mathfrak{p}_i, G_i) Id)$. Then $T(F) \neq 0$ yet $T(\sum_i \sum_{\mathfrak{a}_i} c_i(G_i \mid B_{\mathfrak{a}_i})) = 0$. This contradiction implies that there exists a newform G_i such that $F \sim G_i$. The uniqueness of G_i follows immediately from Theorem 3.6. A similar argument proves assertion (2). □

A consequence of Proposition 3.13 will be that a newform $f \in \mathscr{E}_k(\mathcal{N}, \Psi)$ is an eigenform of the Hecke operator $T_\mathfrak{q}$ for $\mathfrak{q}$ a prime dividing $\mathcal{N}$. We first prove that $T_\mathfrak{q}$ maps $\mathscr{E}_k(\mathcal{N}, \Psi)$ to $\mathscr{E}_k(\mathcal{N}, \Psi)$.

Proposition 3.14. *Let $\mathfrak{q}$ be a prime dividing $\mathcal{N}$. Then $T_\mathfrak{q}$ maps $\mathscr{E}_k(\mathcal{N}, \Psi)$ to $\mathscr{E}_k(\mathcal{N}, \Psi)$.*

Proof. Let $f \in \mathscr{E}_k(\mathcal{N}, \Psi)$. Since $\mathscr{M}_k(\mathcal{N}, \Psi)$ is stable under the action of $T_\mathfrak{q}$ and $\mathscr{M}_k(\mathcal{N}, \Psi) = \mathscr{E}_k(\mathcal{N}, \Psi) \oplus \mathscr{S}_k(\mathcal{N}, \Psi)$, we may write $f \mid T_\mathfrak{q} = g + h$ where $g \in \mathscr{E}_k(\mathcal{N}, \Psi)$ and $h \in \mathscr{S}_k(\mathcal{N}, \Psi)$. We show that $h = 0$. Fix a prime $\mathfrak{p}$ which represents the trivial ideal class modulo $\mathcal{N}\mathfrak{P}_\infty$. By Remark 3.12 and the fact that $T_\mathfrak{p}$ commutes with $T_\mathfrak{q}$, both $f \mid T_\mathfrak{q}$ and g are eigenforms for $T_\mathfrak{p}$ with eigenvalue $1 + N(\mathfrak{p})^{k-1}$. It follows that if $h \neq 0$ then h is an eigenform for $T_\mathfrak{p}$ with eigenvalue equal to $1 + N(\mathfrak{p})^{k-1}$. But this contradicts [13], where it is shown that the $T_\mathfrak{p}$-eigenvalue of h has absolute value less than $2N(\mathfrak{p})^{\frac{k-1}{2}+\frac{1}{5}}$. Therefore $h = 0$ and $f \mid T_\mathfrak{q} \in \mathscr{E}_k(\mathcal{N}, \Psi)$. □

Proposition 3.15. *If E_{ψ_1,ψ_2} is a newform of $\mathscr{E}_k(\mathcal{N}, \Psi)$ then E_{ψ_1,ψ_2} is a eigenform of $T_\mathfrak{p}$ for all primes $\mathfrak{p}$. For every prime $\mathfrak{p}$, the $T_\mathfrak{p}$-eigenvalue of E_{ψ_1,ψ_2} is $C(\mathfrak{p}, E_{\psi_1,\psi_2})$.*

Proof. Proposition 3.3 shows that E_{ψ_1,ψ_2} is an eigenform of $T_\mathfrak{p}$ whenever $\mathfrak{p} \nmid \mathcal{N}$. Let $\mathfrak{q}$ be a prime dividing $\mathcal{N}$. Then $E_{\psi_1,\psi_2} \mid T_\mathfrak{q}$ lies in $\mathscr{E}_k(\mathcal{N}, \Psi)$ and is an eigenform of $T_\mathfrak{p}$ for all primes $\mathfrak{p}$ not dividing $\mathcal{N}$ (as $T_\mathfrak{p}$ and $T_\mathfrak{q}$ commute). By Proposition 3.13 there exists a unique newform to which $E_{\psi_1,\psi_2} \mid T_\mathfrak{q}$ is equivalent. This newform is E_{ψ_1,ψ_2}, since the $T_\mathfrak{p}$-eigenvalues of $E_{\psi_1,\psi_2} \mid T_\mathfrak{q}$ are exactly those of E_{ψ_1,ψ_2} when $\mathfrak{p} \nmid \mathcal{N}$. As $E_{\psi_1,\psi_2} \mid T_\mathfrak{q}$ and E_{ψ_1,ψ_2} are of the same level, they are multiples of one another, proving our assertion. Comparing first coefficients shows that the $T_\mathfrak{q}$-eigenvalue of E_{ψ_1,ψ_2} is $C(\mathfrak{q}, E_{\psi_1,\psi_2})$. □

4. Diagonalizing the space $\mathscr{M}_k(\mathcal{N}, \Psi)$

Consider the space $\mathscr{S}_k(\mathcal{N}, \Psi)$ of Hilbert modular cusp forms of weight k, level $\mathcal{N}$ and Hecke character Ψ. The theory of newforms [1, 8, 16] shows that this space decomposes into a direct sum of common eigenspaces of the Hecke operators $T_\mathfrak{p}$ for all primes $\mathfrak{p} \nmid \mathcal{N}$. The nonzero forms in an eigenspace of dimension one are the newforms of exact level $\mathcal{N}$. The eigenspaces of dimension greater than one are generated by a newform of level $\mathcal{M}$ dividing $\mathcal{N}$ and its shifts by ideals $\mathfrak{a} \mid \mathcal{N}\mathcal{M}^{-1}$. In [11], Pizer considered the space of elliptic cusp forms with trivial character and introduced the operator $C_\mathfrak{q}$ (for $\mathfrak{q}$ a prime dividing $\mathcal{N}$), a diagonalizable operator whose action on the subspace generated by newforms of level $\mathcal{N}$ coincides with that of $T_\mathfrak{q}$. He then showed that the entire space of cusp forms can be diagonalized with respect to the algebra generated by $T_\mathfrak{p}$ (with $\mathfrak{p} \nmid \mathcal{N}$) and $C_\mathfrak{q}$ (with $\mathfrak{q} \mid \mathcal{N}$) and that each of the common eigenspaces is one-dimensional. This result was later generalized to elliptic cusp forms of arbitrary character by Li [9] and to Hilbert modular cusp forms of arbitrary Hecke character by the first author [3].

In this section we will review the construction of the $C_\mathfrak{q}$ operator and consider its action on $\mathscr{E}_k(\mathcal{N}, \Psi)$. In particular we will show that $\mathscr{E}_k(\mathcal{N}, \Psi)$ can be diagonalized with respect to the algebra generated by $T_\mathfrak{p}$ (with $\mathfrak{p} \nmid \mathcal{N}$) and $C_\mathfrak{q}$ (with $\mathfrak{q} \mid \mathcal{N}$) and that each of the common eigenspaces is one-dimensional. As a corollary we will deduce that the entire space $\mathscr{M}_k(\mathcal{N}, \Psi)$ can be diagonalized with respect to the aforementioned algebra and that the corresponding eigenspaces are all one-dimensional.

First let us recall the definition of the $C_\mathfrak{q}$ operator. In the following, if $\psi_\mathcal{Q} \equiv 1$, we will always choose $\Psi_\mathcal{Q} \equiv 1$ to extend it. For a prime $\mathfrak{q} \mid \mathcal{N}$, choose a Hecke character $\Psi_\mathcal{Q}$ extending $\psi_\mathcal{Q}$. The $C_\mathfrak{q}$ operator will depend upon the choice of $\Psi_\mathcal{Q}$, so henceforth we will denote the operator by $C_\mathfrak{q}(\Psi_\mathcal{Q})$. We define $C_\mathfrak{q}(\Psi_\mathcal{Q})$ as follows

$$C_\mathfrak{q}(\Psi_\mathcal{Q}) = \begin{cases} T_\mathfrak{q} \qquad \text{if } \psi \text{ is not a character modulo } \mathcal{N}\mathfrak{q}^{-1} \\[2ex] T_\mathfrak{q} + W_\mathcal{Q}(1) T_\mathfrak{q} W_\mathcal{Q}^{-1}(1) + N(\mathfrak{q})^{\frac{k}{2}-1} W_\mathcal{Q}(1) \\ \qquad \text{if } \psi \text{ is a character modulo } \mathcal{N}\mathfrak{q}^{-1} \text{ and } \mathfrak{q} \parallel \mathcal{N} \\[2ex] T_\mathfrak{q} + W_\mathcal{Q}(\Psi_\mathcal{Q}) T_\mathfrak{q} W_\mathcal{Q}^{-1}(\Psi_\mathcal{Q}) \\ \qquad \text{if } \psi \text{ is a character modulo } \mathcal{N}\mathfrak{q}^{-1} \text{ and } \mathfrak{q}^2 \mid \mathcal{N} \end{cases}$$

Here $W_\mathcal{Q}(\Psi_\mathcal{Q})$ is the Hilbert analogue of the Atkin-Lehner W_Q operator as defined in [14]. It is easy to see that $C_\mathfrak{q}(\Psi_\mathcal{Q})$ is an endomorphism of the space $\mathscr{E}_k(\mathcal{N}, \Psi)$.

Proposition 4.1. *The operator $C_\mathfrak{q}(\Psi_\mathcal{Q})$ commutes with the following operators:*

1. *The Hecke operators $T_\mathfrak{p}$ for all primes $\mathfrak{p} \nmid \mathcal{N}$*
2. *The shift operators $B_\mathcal{L}$ for all integral ideals $\mathcal{L}$ with $\mathfrak{q} \nmid \mathcal{L}$*
3. *The operators $C_{\mathfrak{q}'}(\Psi_{\mathcal{Q}'})$ for all primes $\mathfrak{q}' \mid \mathcal{N}$ with $\mathfrak{q}' \neq \mathfrak{q}$*

Proof. For cuspidal Hilbert modular forms this was proven as Propositions 2.1 and 2.2 of [3]. Essentially identical proofs apply in the context of Eisenstein series. □

Let f be a newform of $\mathscr{M}_k(\mathcal{N}, \Psi)$ having level $\mathcal{N}$ (i.e., f is a newform of $\mathscr{S}_k(\mathcal{N}, \Psi)$ in the sense of [14] or of $\mathscr{E}_k(\mathcal{N}, \Psi)$ in the sense of Section 3). By Proposition 4.1, $f \mid C_{\mathfrak{q}}(\Psi_{\mathcal{Q}})$ is a simultaneous Hecke eigenform for all $T_{\mathfrak{p}}$ with $\mathfrak{p} \nmid \mathcal{N}$ having $C(\mathfrak{p}, f)$ as its $T_{\mathfrak{p}}$-eigenvalue. Because f and $f \mid C_{\mathfrak{q}}(\Psi_{\mathcal{Q}})$ have the same level, there exists a constant $c \in \mathbb{C}$ such that $f \mid C_{\mathfrak{q}}(\Psi_{\mathcal{Q}}) = cf$. In the cuspidal case this constant turns out to be $C(\mathfrak{q}, f)$ (see Proposition 2.6 of [3]). In particular the action of $C_{\mathfrak{q}}(\Psi_{\mathcal{Q}})$ on the space of cuspidal newforms is equal to that of $T_{\mathfrak{q}}$.

Theorem 4.3 will show that the space $\mathscr{E}_k(\mathcal{N}, \Psi)$ can be decomposed into a direct sum of one dimensional eigenspaces of the Hecke operators $T_{\mathfrak{p}}$ for $\mathfrak{p} \nmid \mathcal{N}$ and the operators $C_{\mathfrak{q}}(\Psi_{\mathcal{Q}})$ for $\mathfrak{q} \mid \mathcal{N}$. This was shown for elliptic cusp forms by Li in [9, Theorem 3.6]. In a remark appearing directly after the statement of Theorem 3.6 (on page 222) Li notes that the proof of the main theorem can be extended to the entire space of elliptic modular forms by virtue of the newform theory that Weisinger [17] developed for elliptic modular Eisenstein series. Analogously, Theorem 4.3 follows from the corresponding proof for Hilbert modular cusp forms [3] and the newform theory developed in Section 3. The only result that we have not yet proven and which will be needed for this extension is the following proposition.

Proposition 4.2. *If $E = E_{\psi_1,\psi_2} \in \mathscr{E}_k^{(new)}(\mathcal{N}, \Psi)$ is a newform then $E \mid W_{\mathcal{Q}}(\Psi_{\mathcal{Q}}) = \gamma F$ for some newform $F \in \mathscr{E}_k^{(new)}(\mathcal{N}, \Psi\overline{\Psi}_{\mathcal{Q}}^2)$ and nonzero constant γ. In fact, $F = E_{\psi_1',\psi_2'}$, where $\psi_1' = (\psi_1)_{\mathcal{N}\mathcal{Q}^{-1}}(\overline{\psi}_2)_{\mathcal{Q}}$ and $\psi_2' = (\overline{\psi}_1)_{\mathcal{Q}}(\psi_2)_{\mathcal{N}\mathcal{Q}^{-1}}$.*

Proof. By [14, Proposition 2.4] and Proposition 3.3, we have for every prime $\mathfrak{p} \neq \mathfrak{q}$,

$$E \mid W_{\mathcal{Q}}(\Psi_{\mathcal{Q}}) \mid T_{\mathfrak{p}} = C(\mathfrak{p}, E_{\psi_1',\psi_2'})E \mid W_{\mathcal{Q}}(\Psi_{\mathcal{Q}}).$$

By assertion (1) of Proposition 3.13 there exists a unique newform F of level dividing $\mathcal{M}$ (for some $\mathcal{M} \mid \mathcal{N}$) which is equivalent to $E \mid W_{\mathcal{Q}}(\Psi_{\mathcal{Q}})$. In light of Theorem 3.6 it is clear that $F = E_{\psi_1',\psi_2'}$ (note that $\mathrm{cond}(\psi_1')\mathrm{cond}(\psi_2') = \mathcal{N}$ so that $E_{\psi_1',\psi_2'} \in \mathscr{E}_k^{(new)}(\mathcal{N}, \Psi\overline{\Psi}_{\mathcal{Q}}^2)$). The proposition therefore follows from assertion (2) of Proposition 3.13. □

Theorem 4.3. *For each divisor $\mathfrak{q}$ of $\mathcal{N}$ let $\Psi_{\mathcal{Q}}$ be a Hecke character extending $\psi_{\mathcal{Q}}$. Then the space $\mathscr{E}_k(\mathcal{N}, \Psi)$ can be decomposed into a direct sum of common eigenspaces of $\{T_{\mathfrak{p}} : \mathfrak{p} \nmid \mathcal{N}\}$ and $\{C_{\mathfrak{q}}(\Psi_{\mathcal{Q}}) : \mathfrak{q} \mid \mathcal{N}\}$, each of dimension one. Each common eigenspace is spanned by a form h with Dirichlet series*

$$D(s, h) = \sum_{\mathfrak{m} \subset \mathcal{O}} C(\mathfrak{m}, h)N(\mathfrak{m})^{-s}$$

in which $C(\mathcal{O}, h) = 1$, $h \mid T_{\mathfrak{p}} = C(\mathfrak{p}, h)h$ for all $\mathfrak{p} \nmid \mathcal{N}$, and $h \mid C_{\mathfrak{q}}(\Psi_{\mathcal{Q}}) = C(\mathfrak{q}, h)h$ for all $\mathfrak{q} \mid \mathcal{N}$. In addition, for such h we have $C(\mathfrak{m}\mathfrak{n}, h) = C(\mathfrak{m}, h)C(\mathfrak{n}, h)$ for $(\mathfrak{m}, \mathfrak{n}) = 1$.

Combining Theorem 4.3 with Theorem 2.7 of [3] gives us the following.

Corollary 4.4. *For each prime divisor $\mathfrak{q}$ of $\mathcal{N}$ let $\Psi_{\mathcal{Q}}$ be a Hecke character extending $\psi_{\mathcal{Q}}$. The space $\mathscr{M}_k(\mathcal{N},\Psi)$ may be decomposed into a direct sum of common eigenspaces, each of dimension one, for the algebra generated by the Hecke operators $\{T_\mathfrak{p} : \mathfrak{p} \nmid \mathcal{N}\}$ and the operators $\{C_\mathfrak{q}(\Psi_{\mathcal{Q}}) : \mathfrak{q} \mid \mathcal{N}\}$.*

Proof. We need only show that if $g, h \in \mathscr{M}_k(\mathcal{N},\Psi)$ are simultaneous eigenforms for all of the $T_\mathfrak{p}$ and $C_\mathfrak{q}$ operators with $g \in \mathscr{E}_k(\mathcal{N},\Psi)$ and $h \in \mathscr{S}_k(\mathcal{N},\Psi)$ then the eigenspaces to which g and h belong are distinct. This follows from standard estimates on the absolute values of $|C(\mathfrak{p}, g)|$ and $|C(\mathfrak{p}, h)|$ (cf. the proof of Proposition 3.9). ☐

5. Twists of Hilbert Eisenstein series

In this section we study the behavior of Hilbert Eisenstein series under twists by a Hecke character. By virtue of Proposition 3.11 it suffices to consider twists of newforms by Hecke characters.

Definition 5.1. Let $f \in \mathscr{M}_k(\mathcal{N},\Psi)$ and Φ be a Hecke character. We define the twist of f by Φ, denoted f_Φ, by

$$f_\Phi(x) = \tau(\overline{\Phi})^{-1}\Phi(\det x) \sum_{r \in \mathfrak{f}_\Phi^{-1}\mathfrak{d}^{-1}/\mathfrak{d}^{-1}} \overline{\Phi}_\infty(r)\overline{\Phi}^*(r\mathfrak{f}_\Phi\,\mathfrak{d}) f \mid \left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right)_0 (x),$$

where $\tau(\overline{\Phi})$ is the Gauss sum associated to $\overline{\Phi}$ defined in (9.31) of [15] and the subscript 0 denotes the projection onto the nonarchimedean part.

Proposition 5.2. *If $f \in \mathscr{M}_k(\mathcal{N},\Psi)$ and Φ is a Hecke character then*

(1) $f_\Phi \in \mathscr{M}_k(\mathcal{L}, \Psi\Phi^2)$ where $\mathcal{L}$ is the least common multiple of $\mathcal{N}$, $\mathfrak{f}_\Phi^2$ and $\mathfrak{f}_\Phi\mathfrak{f}_\Psi$.
(2) $C(\mathfrak{m}, f_\Phi) = \Phi^*(\mathfrak{m})C(\mathfrak{m}, f)$ for all integral ideals $\mathfrak{m}$.

Proof. This is the final assertion of Proposition 9.7 of [15]. ☐

The following proposition is trivial to verify using the action of the Hecke operators $T_\mathfrak{p}$ on $\mathscr{E}_k(\mathcal{N},\Psi)$.

Proposition 5.3. *Let Φ be a Hecke character of conductor $\mathcal{M}$ and let $\mathfrak{p}$ be a prime not dividing $\mathcal{MN}$. Then for $f \in \mathscr{E}_k(\mathcal{N},\Psi)$ we have $f_\Phi \mid T_\mathfrak{p} = \Phi^*(\mathfrak{p})(f \mid T_\mathfrak{p})_\Phi$.*

Proposition 5.3 shows that the twist of a simultaneous Hecke eigenform is again a simultaneous Hecke eigenform. It is therefore natural to ask when the twist of a newform will be a newform. Note that it clearly suffices to consider twists by a Hecke character Φ whose conductor is a power of a prime $\mathfrak{p}$. Henceforth we assume that $\operatorname{cond}(\Phi) = \mathfrak{p}^\beta$ where $\beta \geq 1$.

Suppose that f is a cuspidal newform of $\mathscr{S}_k(\mathcal{N},\Psi)$ and Φ is a Hecke character as above. If the conductor of Φ is coprime to $\mathcal{N}$ then it is well known that f_Φ is a newform of

$\mathscr{S}_k(\mathcal{N}\mathfrak{f}_\Phi^2, \Psi\Phi^2)$ ([14, Theorem 5.5]). If the conductor of Φ is not coprime to $\mathcal{N}$ then the question of whether or not f_Φ is a newform is more subtle. In the latter situation it is known that there exists a newform g of suitable level such that $f_\Phi = g - g \mid T_\mathfrak{p} \mid B_\mathfrak{p}$ ([14, Theorem 5.8]). Thus f_Φ is a newform if and only if $g \mid T_\mathfrak{p} = 0$.

The explicit nature of the newform theory developed in Section 3 (in particular the fact that we may easily compute the $\mathfrak{m}$-th coefficient of a Hilbert Eisenstein newform) eliminates many of the difficulties present in the cuspidal case. We begin by proving an analogue of [14, Theorem 5.8].

Proposition 5.4. *Let $f = E_{\psi_1,\psi_2} \in \mathscr{E}_k(\mathcal{N}, \Psi)$ be a newform and Φ be a nontrivial Hecke character whose conductor is a power of a prime $\mathfrak{p}$. Then there exists a newform $g \in \mathscr{E}_k(\mathcal{L}, \Psi\Phi^2)$ such that $f_\Phi = g - g \mid T_\mathfrak{p} \mid B_\mathfrak{p}$. Here $\mathcal{L} = \operatorname{cond}(\psi_1\Phi^*)\operatorname{cond}(\psi_2\Phi^*)$.*

Proof. Let $\mathfrak{m}$ be an integral ideal. If $\mathfrak{p} \mid \mathfrak{m}$ then $C(\mathfrak{m}, f_\Phi) = 0$ by Proposition 5.2. If $\mathfrak{p} \nmid \mathfrak{m}$ then

$$\begin{aligned} C(\mathfrak{m}, f_\Phi) = \Phi^*(\mathfrak{m})C(\mathfrak{m}, f) &= \Phi^*(\mathfrak{m})\sum_{\mathfrak{r}\mid\mathfrak{m}} \psi_1(\mathfrak{m}\mathfrak{r}^{-1})\psi_2(\mathfrak{r})N(\mathfrak{r})^{k-1} \\ &= \sum_{\mathfrak{r}\mid\mathfrak{m}} (\psi_1\Phi^*)(\mathfrak{m}\mathfrak{r}^{-1})(\psi_2\Phi^*)(\mathfrak{r})N(\mathfrak{r})^{k-1} \\ &= C(\mathfrak{m}, E_{\psi_1\Phi^*,\psi_2\Phi^*}) \end{aligned}$$

It follows that $f_\Phi = E_{\psi_1\Phi^*,\psi_2\Phi^*} - E_{\psi_1\Phi^*,\psi_2\Phi^*} \mid T_\mathfrak{p} \mid B_\mathfrak{p}$. $\square$

It is now easy to show that when the conductor of Φ is coprime to the level of f, f_Φ will be a newform.

Corollary 5.5. *Let $f \in \mathscr{E}_k(\mathcal{N}, \Psi)$ be a newform and Φ be a nontrivial $\mathfrak{p}$-primary Hecke character. If $(\mathfrak{p}, \mathcal{N}) = 1$ then f_Φ is a newform.*

Proof. Write $f = E_{\psi_1,\psi_2}$. By Proposition 5.4 (and its proof) it suffices to show that $g \mid T_\mathfrak{p} = 0$ where $g = E_{\psi_1\Phi^*,\psi_2\Phi^*}$. Here g is a newform of level $\mathcal{L} = \operatorname{cond}(\psi_1\Phi^*)\operatorname{cond}(\psi_2\Phi^*)$. Note that $\mathfrak{p} \mid \mathcal{L}$ so that $g \mid T_\mathfrak{p} = C(\mathfrak{p}, g)g$. As $C(\mathfrak{p}, g) = (\psi_1\Phi^*)(\mathfrak{p}) + (\psi_2\Phi^*)(\mathfrak{p})N(\mathfrak{p})^{k-1}$ it suffices to show that $\mathfrak{p}$ divides the conductor of $(\psi_i\Phi^*)$ for $i = 1, 2$. But $\operatorname{cond}(\psi_i\Phi^*) = \operatorname{cond}(\psi_i)\operatorname{cond}(\Phi^*)$ because $\operatorname{cond}(\psi_i) \mid \mathcal{N}$ and $(\mathfrak{p}, \mathcal{N}) = 1$. Therefore $\mathfrak{p} \mid \operatorname{cond}(\psi_i\Phi^*)$ and we're done. $\square$

Corollary 5.6. *Let $f \in \mathscr{E}_k(\mathcal{N}, \Psi)$ be a newform with $\mathfrak{p} \nmid \operatorname{cond}(\Psi)$ and let Φ be a nontrivial $\mathfrak{p}$-primary Hecke character such that $2 \cdot ord_\mathfrak{p}(\operatorname{cond}(\Phi)) \neq ord_\mathfrak{p}(\mathcal{N})$. Then f_Φ is a newform.*

Proof. Write $f = E_{\psi_1,\psi_2}$ and denote by ψ_1' (respectively ψ_2') the character modulo $\mathcal{N}$ induced by ψ_1 (respectively ψ_2). By the proof of Proposition 5.4 it suffices to prove that $g \mid T_\mathfrak{p} = 0$ where $g = E_{\psi_1\Phi^*,\psi_2\Phi^*}$. Because g has $C(\mathfrak{p}, g)$ as its eigenvalue for the Hecke operator $T_\mathfrak{p}$, it suffices to show that $C(\mathfrak{p}, g) = 0$. Because $\psi_1'\psi_2' = \Psi^*$ and $\operatorname{cond}(\psi_1)\operatorname{cond}(\psi_2) = \mathcal{N}$, our hypothesis that $\mathfrak{p} \nmid \operatorname{cond}(\Psi)$ implies that $\operatorname{ord}_\mathfrak{p}(\operatorname{cond}(\psi_1)) = \operatorname{ord}_\mathfrak{p}(\operatorname{cond}(\psi_2)) = \frac{1}{2}\operatorname{ord}_\mathfrak{p}(\mathcal{N})$.

As $\mathrm{ord}_{\mathfrak{p}}(\mathrm{cond}(\Phi)) \neq \frac{1}{2}\mathrm{ord}_{\mathfrak{p}}(\mathcal{N})$ we see that both $\mathrm{ord}_{\mathfrak{p}}(\mathrm{cond}(\psi_1\Phi^*))$ and $\mathrm{ord}_{\mathfrak{p}}(\mathrm{cond}(\psi_2\Phi^*))$ are nonzero. In particular $(\psi_1\Phi^*)(\mathfrak{p}) = 0 = (\psi_2\Phi^*)(\mathfrak{p})$. Then $C(\mathfrak{p}, g) = (\psi_1\Phi^*)(\mathfrak{p}) + (\psi_2\Phi^*)(\mathfrak{p})N(\mathfrak{p})^{k-1} = 0$ and we're done. ☐

Definition 5.7. Let $\mathfrak{q}$ be a prime dividing $\mathcal{N}$. In analogy with [2] we say that a newform $f \in \mathscr{E}_k(\mathcal{N}, \Psi)$ is $\mathfrak{q}$-primitive if f is not the twist of any newform g of level $\mathcal{N}'$ where $\mathcal{N}'$ is a proper divisor of $\mathcal{N}$ by a Hecke character whose conductor is a power of $\mathfrak{q}$. It is clear that f is $\mathfrak{q}$-primitive whenever $C(\mathfrak{q}, f) \neq 0$.

The following are 'Eisenstein' analogues of Propositions 6.5 and 6.6 of [14].

Proposition 5.8. *If $f \in \mathscr{E}_k(\mathcal{N}, \Psi)$ is a $\mathfrak{q}$-primitive newform and $C(\mathfrak{q}, f) = 0$, then all twists of f by $\mathfrak{q}$-primary Hecke characters Φ are newforms with level divisible by $\mathcal{N}$.*

Proof. Let Φ be a $\mathfrak{q}$-primary Hecke character and write $f = E_{\psi_1,\psi_2}$. By Proposition 5.4 there exists a newform g of level $\mathcal{L} = \mathrm{cond}(\psi_1\Phi^*)\,\mathrm{cond}(\psi_2\Phi^*)$ such that $f_\Phi = g - C(\mathfrak{q}, g)g \mid B_{\mathfrak{q}}$. We begin by supposing that f_Φ is not a newform or equivalently, that $C(\mathfrak{q}, g) \neq 0$. Because $C(\mathfrak{q}, f) = 0$ we see that $f = f_{\Phi\overline{\Phi}} = g_{\overline{\Phi}}$. By hypothesis f is $\mathfrak{q}$-primitive, so it must be the case that $\mathrm{ord}_{\mathfrak{q}}(\mathcal{L}) \geq \mathrm{ord}_{\mathfrak{q}}(\mathcal{N})$. This is equivalent to the inequality $\mathrm{ord}_{\mathfrak{q}}(\mathrm{cond}(\psi_1\Phi^*)) + \mathrm{ord}_{\mathfrak{q}}(\mathrm{cond}(\psi_2\Phi^*)) \geq \mathrm{ord}_{\mathfrak{q}}(\mathrm{cond}(\psi_1)) + \mathrm{ord}_{\mathfrak{q}}(\mathrm{cond}(\psi_2))$. We have assumed that $C(\mathfrak{q}, f) = \psi_1(\mathfrak{q}) + \psi_2(\mathfrak{q})N(\mathfrak{q})^{k-1} = 0$ so $\mathrm{ord}_{\mathfrak{q}}(\mathrm{cond}(\psi_i)) > 0$ for $i = 1, 2$. Similarly, $C(\mathfrak{q}, g) = (\psi_1\Phi^*)(\mathfrak{q}) + (\psi_2\Phi^*)(\mathfrak{q})N(\mathfrak{p})^{k-1} \neq 0$ so without loss of generality we may assume that $\mathrm{ord}_{\mathfrak{q}}(\mathrm{cond}(\psi_1\Phi^*)) = 0$. Using the fact that $\mathrm{ord}_{\mathfrak{q}}(\mathrm{cond}(\psi_2\Phi^*)) \leq \max\{\mathrm{ord}_{\mathfrak{q}}(\mathrm{cond}(\Phi^*), \mathrm{ord}_{\mathfrak{q}}(\mathrm{cond}(\psi_2)\}$ we see that it must be the case that $\mathrm{ord}_{\mathfrak{q}}(\mathrm{cond}(\Phi^*)) > \mathrm{ord}_{\mathfrak{q}}(\mathrm{cond}(\psi_1)) + \mathrm{ord}_{\mathfrak{q}}(\mathrm{cond}(\psi_2))$. In particular $\mathrm{ord}_{\mathfrak{q}}(\mathrm{cond}(\Phi^*)) > \mathrm{ord}_{\mathfrak{q}}(\mathrm{cond}(\psi_1))$. But if this inequality were to hold it could never be the case that $\mathrm{ord}_{\mathfrak{q}}(\mathrm{cond}(\psi_1\Phi^*)) = 0$. This is a contradiction and shows that f_Φ is a newform. The level of f_Φ is divisible by $\mathcal{N}$ because otherwise the fact that $f = (f_\Phi)_{\overline{\Phi}}$ would contradict f being $\mathfrak{q}$-primitive. ☐

Proposition 5.9. *If all twists of f by Hecke characters Φ with $\mathfrak{q}$-primary conductors are newforms, then f is the twist of a $\mathfrak{q}$-primitive newform.*

Proof. The proof of this proposition is exactly analogous to the proof of Proposition 6.6 of [14]. ☐

6. Obstruction sets

In this section we use the newform theory developed in Section 3 in order to provide a framework for studying Eisenstein series $f \in \mathscr{E}_k(\mathcal{N}, \Psi)$ which are Hecke eigenforms for some, but not all, of the Hecke operators $T_{\mathfrak{p}}$. Such Eisenstein series arise naturally in the theory of quadratic forms when one considers the genus theta series associated to a positive definite quadratic form (cf. [7]). In particular we will relate the density of the set of primes $\mathfrak{p}$ for which f is an eigenform for the Hecke operator $T_{\mathfrak{p}}$ to the number of distinct classes of newforms of which f is a sum.

Definition 6.1. Let $S(f)$ be the set of finite primes of K for which f is not a $T_{\mathfrak{p}}$-eigenform. We call $S(f)$ the obstruction set of f.

It is immediate from the definition that $S(f)$ is empty if and only if f is a simultaneous eigenform for all Hecke operators $T_{\mathfrak{p}}$. Therefore the obstruction set of f (or rather its density relative to the set of primes of K) allows us to quantify how close f is to being a simultaneous eigenform the Hecke operators $T_{\mathfrak{p}}$.

We begin by proving a few basic properties of the obstruction set of an Eisenstein series.

Proposition 6.2. *Let notation be as above. The following are equivalent.*

1. *$S(f)$ is finite.*
2. *There is a newform $g \in \mathscr{E}_k(\mathcal{M}, \Psi)$ such that f is a linear combination of g and its shifts.*
3. *f is a simultaneous Hecke eigenform for all $T_{\mathfrak{p}}$ with $\mathfrak{p} \nmid \mathcal{N}$.*
4. *$S(f)$ has density zero relative to the set of all primes of K.*

Proof. We first show that $S(f)$ is finite if and only if there is a newform $g \in \mathscr{E}_k(\mathcal{M}, \Psi)$ (for some $\mathcal{M} \mid \mathcal{N}$) such that f is a linear combination of g and its shifts.

It is clear that any linear combination of shifts of a single newform will be a Hecke eigenform for almost all Hecke operators $T_{\mathfrak{p}}$. It therefore suffices to show that if $S(f)$ is finite then f is such a linear combination. To that end, write

$$f = \sum_{i=1}^{r} \alpha_i \left(g_i \mid B_{\mathfrak{a}_i}\right),$$

where the α_i are nonzero complex numbers, each g_i is a newform of level $\mathcal{M}_i$ dividing $\mathcal{N}$ and $\mathfrak{a}_i$ is an integral ideals dividing $\mathcal{N}\mathcal{M}_i^{-1}$. Because $S(f)$ is finite, f must be a Hecke eigenform for almost all primes $\mathfrak{p}$. It follows that the g_i have the same $T_{\mathfrak{p}}$-eigenvalues for almost all primes $\mathfrak{p}$, hence are equal by Theorem 3.6.

If f is a linear combination of a newform g and its shifts then f must be a simultaneous Hecke eigenform for all primes $\mathfrak{p} \nmid \mathcal{N}$ because $T_{\mathfrak{p}}$ and $B_{\mathfrak{r}}$ commute whenever $\mathfrak{p} \nmid \mathfrak{r}$. The converse follows from Proposition 3.13.

If $S(f)$ is finite then it must have density zero relative to the set of all primes of K, so all that remains is to show is that if $S(f)$ has density zero then $S(f)$ is finite. Suppose therefore that $S(f)$ has density zero but contains infinitely many primes of K. We will derive a contradiction. Write $f = \sum_{i=1}^{r} \alpha_i \left(g_i \mid B_{\mathfrak{a}_i}\right)$ as in the proof of the second assertion. Let $\mathfrak{p}_0$ be an element of $S(f)$ which is coprime to $\mathcal{N}$. In particular $\mathfrak{p}_0$ is coprime to each of the ideals $\mathfrak{a}_i$. Because f is not a $T_{\mathfrak{p}_0}$-eigenform, there exist newforms g_i and g_j (appearing in the decomposition of f) such that $C(\mathfrak{p}_0, g_i) \neq C(\mathfrak{p}_0, g_j)$. If we write $g_i = E_{\chi_1,\chi_2}$ and $g_j = E_{\psi_1,\psi_2}$ then we see that $\chi_1(\mathfrak{p}_0) + \chi_2(\mathfrak{p}_0)N(\mathfrak{p}_0)^{k-1} \neq \psi_1(\mathfrak{p}_0) + \psi_2(\mathfrak{p}_0)N(\mathfrak{p}_0)^{k-1}$. It follows that $\chi_1(\mathfrak{p}_0) \neq \psi_1(\mathfrak{p}_0)$ and $\chi_2(\mathfrak{p}_0) \neq \psi_2(\mathfrak{p}_0)$. By the Chebotarev Density Theorem, the set of primes $\mathfrak{q}$ such that $\chi_1(\mathfrak{q}) \neq \psi_1(\mathfrak{q})$ and $\chi_2(\mathfrak{q}) \neq \psi_2(\mathfrak{q})$ has positive density. For such a prime $\mathfrak{q}$, $\chi_1(\mathfrak{q}) + \chi_2(\mathfrak{q})N(\mathfrak{q})^{k-1} \neq \psi_1(\mathfrak{q}) + \psi_2(\mathfrak{q})N(\mathfrak{q})^{k-1}$ and hence $C(\mathfrak{q}, g_i) \neq C(\mathfrak{q}, g_j)$, so $\mathfrak{q} \in S(f)$. This contradicts our assumption that $S(f)$ has density zero. □

An argument similar to the one used in the proof of Proposition 6.2 can be used to show that the set $S(f)$ has density 1 relative to the set of all primes of K if and only if $S(f)$ is equal to the set of all primes of K. This is unnecessary however, as we saw in Remark 3.12 that every element of $\mathscr{E}_k(\mathcal{N}, \Psi)$ is a $T_\mathfrak{p}$-eigenform for a set of primes $\mathfrak{p}$ having positive density relative to the set of all primes of K. Therefore the density of $S(f)$ relative to the set of all primes of K is always strictly less than 1.

Proposition 6.3. *Let m be the number of ideal classes modulo $\mathcal{N}\mathfrak{P}_\infty$. If $f \in \mathscr{E}_k(\mathcal{N}, \Psi)$ then $\delta(S(f))$ is an integer multiple of $\frac{1}{m}$.*

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be distinct primes which represent the m ideal classes modulo $\mathcal{N}\mathfrak{P}_\infty$. We may assume that $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ are indexed so that f is a $T_{\mathfrak{p}_i}$-eigenform for $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ but not for $\mathfrak{p}_{s+1}, \ldots, \mathfrak{p}_m$. We note that by Remark 3.12, $s \geq 1$. As above, write $f = \sum_{i=1}^r \alpha_i \,(g_i \mid B_{\mathfrak{a}_i})$ where each g_i is of the form $E_{\chi_1^{(i)}, \chi_2^{(i)}}$. We have seen that f being a $T_{\mathfrak{p}_i}$-eigenform is equivalent to $C(\mathfrak{p}_i, g_1) = \cdots = C(\mathfrak{p}_i, g_r)$ and hence to $\mathfrak{p}_i$ being a solution to the system

$$\chi_1^{(1)}(\mathfrak{p}_i) = \cdots = \chi_1^{(r)}(\mathfrak{p}_i) \ .$$

If $\mathfrak{p}_i$ is a solution to this system, then so too is any prime which represents the same ideal class modulo $\mathcal{N}\mathfrak{P}_\infty$ as $\mathfrak{p}_i$. As the primes of K are equidistributed amongst the ideal classes modulo $\mathcal{N}\mathfrak{P}_\infty$, we are done. □

Not every value $\frac{x}{m}$ is actually obtained as the density $\delta(S(f))$ of some $f \in \mathscr{E}_k(\mathcal{N}, \Psi)$. In Proposition 6.4 for instance, we show that if $x \neq 0$ then $x \geq \lceil \frac{m}{2} \rceil$.

Proposition 6.4. *If $\delta(S(f)) \leq \frac{1}{2}$ then f is a simultaneous Hecke eigenform for all $T_\mathfrak{p}$ with $\mathfrak{p} \nmid \mathcal{N}$.*

Proof. Write f as a linear combination $f = \sum_{i=1}^r \alpha_i \,(g_i \mid B_{\mathfrak{a}_i})$ where each newform g_i is of the form $E_{\chi_1^{(i)}, \chi_2^{(i)}}$. Let $1 \leq i < j \leq r$ and consider the characters $\psi_1 := \chi_1^{(i)}\overline{\chi}_1^{(j)}$, $\psi_2 := \chi_2^{(i)}\overline{\chi}_2^{(j)}$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be distinct primes representing the ideal classes modulo $\mathcal{N}\mathfrak{P}_\infty$, indexed so that f is a $T_{\mathfrak{p}_i}$-eigenform for $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ but not for $\mathfrak{p}_{s+1}, \ldots, \mathfrak{p}_m$. Our hypotheses imply that $2s > m$. The proof of Proposition 6.3 shows that $\psi_1(\mathfrak{p}_\ell) = \psi_2(\mathfrak{p}_\ell) = 1$ for $1 \leq \ell \leq s$ (equivalently, f is a $T_{\mathfrak{p}_\ell}$-eigenform for ℓ in this range). It follows that neither $\sum_{k=1}^m \psi_1(\mathfrak{p}_k)$ nor $\sum_{k=1}^m \psi_2(\mathfrak{p}_k)$ is zero. As both ψ_1 and ψ_2 are characters of a finite abelian group, they must be trivial. Therefore for all $1 \leq i, j \leq r$ we have $\chi_1^{(i)} = \chi_1^{(j)}$ and $\chi_2^{(i)} = \chi_2^{(j)}$. If $\mathfrak{p}$ is a prime not dividing $\mathcal{N}$ then the $T_\mathfrak{p}$-eigenvalue of g_i is $\chi_1^{(i)}(\mathfrak{p}) + \chi_2^{(i)}(\mathfrak{p})N(\mathfrak{p})^{k-1}$, hence all of the newforms g_i have equal $T_\mathfrak{p}$-eigenvalues for almost all primes $\mathfrak{p}$. All of the g_i are thus equal by Theorem 3.6. The proof now follows from Proposition 6.2. □

Write $f \in \mathscr{E}_k(\mathcal{N}, \Psi)$ as a linear combination $f = \sum_{i=1}^{r} \alpha_i \,(g_i \mid B_{\mathfrak{a}_i})$ where each g_i is a newform of level $\mathcal{M}_i$ dividing $\mathcal{N}$ and $\mathfrak{a}_i \mid \mathcal{N}\mathcal{M}_i^{-1}$. Note that the g_i are not necessarily distinct.

Definition 6.5. We say that f is a sum of t *classes* of newforms if there are t distinct newforms (each of level dividing $\mathcal{N}$) $E_{\chi_1^{(1)},\chi_2^{(1)}}, \dots, E_{\chi_1^{(t)},\chi_2^{(t)}}$ such that all of the g_i appearing in the above decomposition of f are equal to $E_{\chi_1^{(i)},\chi_2^{(i)}}$ for some $i \leq t$.

By Proposition 6.2, f is a simultaneous eigenform for all of the Hecke operators $T_{\mathfrak{p}}$ (with $\mathfrak{p} \nmid \mathcal{N}$) if and only if f is a sum of 1 class of newforms. In this section we investigate the relationship between the number of classes of newforms of which f is a sum and the density of the obstruction set $S(f)$.

Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ represent the m ideal classes modulo $\mathcal{N}\mathfrak{P}_\infty$ and $G(f)$ denote the subgroup of the ideal class group modulo $\mathcal{N}\mathfrak{P}_\infty$ (which we denote by G) whose elements are the solutions to the system

$$\chi_1^{(1)}(\mathfrak{p}_i) = \cdots = \chi_1^{(r)}(\mathfrak{p}_i).$$

As f is a $T_{\mathfrak{p}_i}$-eigenform if and only if $\mathfrak{p}_i$ is a solution to the above system, we see that if f is a $T_{\mathfrak{p}_i}$-eigenform then f is a $T_{\mathfrak{p}}$-eigenform for every prime $\mathfrak{p}$ which represents the same class modulo $\mathcal{N}\mathfrak{P}_\infty$ as $\mathfrak{p}_i$. It is therefore easy to see that $\delta(S(f)) = (1 - \frac{|G(f)|}{m})$, where m is the size of the ideal class group modulo $\mathcal{N}\mathfrak{P}_\infty$.

We first note that if $m = 1$, or equivalently, the ideal class group modulo $\mathcal{N}\mathfrak{P}_\infty$ is trivial, then there exists only a single class of newforms. In this case $\delta(S(f)) = 0$. Suppose now that the ideal class group modulo $\mathcal{N}\mathfrak{P}_\infty$ is of prime cardinality q. That $G(f)$ is a subgroup of G implies that either $G(f)$ is trivial or $G(f) = G$. In the former case $\delta(S(f)) = 1 - \frac{1}{q}$. In the latter case $\delta(S(f)) = 1 - \frac{q}{q} = 0$; i.e. f is a simultaneous eigenform for all $T_{\mathfrak{p}}$ with $\mathfrak{p} \nmid \mathcal{N}$ and consequently is a sum of 1 class of newforms. Having dispensed with these cases, we henceforth assume that $\#G = m$ is composite.

Theorem 6.6. *Suppose that $\delta(S(f)) = \frac{x}{m}$ for some integer $0 \leq x < m$. Then f is a linear combination of at most $\lfloor \frac{m}{m-x} \rfloor$ classes of newforms.*

Proof. By hypothesis $\delta(S(f)) = \frac{x}{m} = 1 - \frac{\#G(f)}{m}$, hence $\#G(f) = m - x$. Then the index of $G(f)$ in G is $\frac{m}{m-x}$. Observe that all of the characters $\chi_1^{(i)}$ coincide when restricted to $G(f)$. Our result now follows from the fact that if G is a finite abelian group and H a subgroup, then there are $[G : H]$ ways to extend a character on H to a character on G. □

Corollary 6.7. *Suppose that $\delta(S(f)) = \frac{1}{2}$. Then f is a linear combination of 2 classes of newforms.*

Proof. This follows immediately from Theorem 6.6. □

Remark 6.8. Examples of Eisenstein series satisfying the hypothesis of Corollary 6.7 are given by the genus theta series of an even, positive definite matrix of prime level. See [7].

We now prove a "converse" to Theorem 6.6. We will assume that f is a sum of t classes of newforms and prove upper and lower bounds for $\delta(S(f))$. These bounds are of interest due to their uniformity; that is, they apply not to the obstruction set of f but also to that of any other element of $\mathscr{E}_k(\mathcal{N},\Psi)$ which is also a sum of t classes of newforms. The upper bound will depend on the smallest prime divisor of m, which we will denote by q.

Theorem 6.9. *Suppose that f is a linear combination of t classes of newforms. Then*

$$\delta(S(f)) \in (\frac{x_0}{m}, \frac{x_1}{m})$$

where $x_0 = m(1-\frac{1}{t})$ and $x_1 = m - \frac{q^{t-1}}{m^{t-2}}$.

Proof. For the lower bound it suffices to show that $\delta(S(f)) > \frac{t-1}{t}$. Write $\delta(S(f)) = \frac{x}{m}$. It follows from Theorem 6.6 that $t \leq \frac{m}{m-x}$. Equivalently, $x \geq \frac{m(t-1)}{t}$. It is now clear that $\delta(S(f)) = \frac{x}{m} \geq \frac{t-1}{t}$.

For the upper bound it suffices to show that $\delta(S(f)) \leq 1 - \left(\frac{q}{m}\right)^{t-1}$. Recall that $\delta(S(f)) = 1 - \frac{g}{m}$ where $g = \#G(f)$ is the number of solutions (in G) to the system

$$\chi_1^{(1)}(\mathfrak{p}_i) = \cdots = \chi_1^{(r)}(\mathfrak{p}_i).$$

We first note that not all of the characters $\chi_1^{(i)}$ are necessarily distinct. For instance, if multiple (distinct) shifts of the same newform appear as summands in the decomposition $f = \sum_{i=1}^{r} \alpha_i \, (g_i \mid B_{\mathfrak{a}_i})$, then the corresponding character will appear in the system with the same multiplicity. It therefore suffices to assume that $r = t$.

Note also that the solution set of this system is unchanged if we fix a character $\chi : G \to \mathbb{C}^\times$ and multiply each of the characters appearing in the system by χ. Letting χ be the inverse of $\chi_1^{(1)}$ yields

$$\chi_0(\mathfrak{p}_i) = \left(\chi\chi_1^{(2)}\right)(\mathfrak{p}_i) = \cdots = \left(\chi\chi_1^{(t)}\right)(\mathfrak{p}_i).$$

Consequently, $G(f) = \bigcap_{i=2}^{t} \ker\left(\chi\chi_1^{(i)}\right)$. For each i, $\ker\left(\chi\chi_1^{(i)}\right)$ is a subgroup of G, hence the index $[G : \ker\left(\chi\chi_1^{(i)}\right)]$ of $\ker\left(\chi\chi_1^{(i)}\right)$ in G is less than or equal to $\frac{m}{q}$. Then

$$[G : G(f)] \leq \prod_{i=2}^{t} [G : \ker\left(\chi\chi_1^{(i)}\right)] \leq \left(\frac{m}{q}\right)^{t-1},$$

hence $\frac{m}{g} \leq \left(\frac{m}{q}\right)^{t-1}$. It now follows that $\delta(S(f)) = 1 - \frac{g}{m} \leq 1 - \left(\frac{q}{m}\right)^{t-1}$. ∎

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