THE SPECTRAL GEOMETRY OF ARITHMETIC HYPERBOLIC 3-MANIFOLDS

BENJAMIN LINOWITZ

These notes are based on a series of lectures given by the author at the Universidad Nacional de Córdoba - Argentina during the fall of 2015. The purpose of the notes is to introduce the reader to the spectral geometry of arithmetic hyperbolic 3-manifolds and the number theoretic techniques which arise in their study. No previous knowledge of arithmetic manifolds is assumed, and little prerequisite knowledge on the part of the reader is required beyond the rudiments of hyperbolic geometry and algebraic number theory.

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Date: February 23, 2016.

1. Introduction

These notes are based on a series of lectures given at the Universidad Nacional de Córdoba - Argentina during the fall of 2015. Their goal is to introduce the reader, in as gentle a manner as possible, to arithmetic hyperbolic 3-manifolds and their spectral theory. The focal point of the text is the manner in which the geometry of arithmetic hyperbolic 3-manifolds can be studied by means of powerful techniques from algebra and number theory, and conversely the manner in which the algebraic and number theoretic invariants of an arithmetic hyperbolic 3-manifold are reflected in its geometry.

The prerequisites for these notes were intentionally kept to a minimum, and for the most part we develop the necessary tools as they arise. A basic knowledge of hyperbolic geometry would of course be helpful given that the purpose of the notes is to study the geometry of certain hyperbolic manifolds, though is strictly speaking not necessary as all of the background needed is reviewed in Section 2. One of the central themes of these notes is that arithmetic hyperbolic 3-manifolds can be viewed as arising naturally in algebraic number theory. In particular we will associate to every hyperbolic 3-manifold of finite volume a number field (called the *invariant trace field*) and a quaternion algebra defined over said number field (the *invariant quaternion algebra*). These objects will be studied extensively throughout the notes, and as such we assume that the reader is familiar with the basic concepts of algebraic number theory; i.e., the arithmetic of prime ideals in number fields, the structure of the group of units of a number field and the theory of local fields. We do not assume any previous exposure to quaternion algebras, and will review all of the needed theory in Section 3.

We now discuss the content of these notes. As was mentioned above, Section 2 treats the terminology and background from hyperbolic geometry that we will be making use of. As the reader is assumed to be familiar with much of this material, our treatment is necessarily terse.

In Section 3 we introduce quaternion algebras and the techniques from non-commutative algebra which are crucial to their study (e.g., Wedderburn's Structure Theorem, the Skolem-Noether Theorem, etc). We will discuss the structure of quaternion division algebras over number fields and their completions, culminating with a statement of the classification of quaternion algebras over number fields in terms of their ramification data.

Section 4 defines the invariant trace field and invariant quaternion algebra of a hyperbolic 3-manifold of finite volume, perhaps the two most important arithmetic invariants discussed in the text. We show that these are commensurability class invariants. As a first application of this theory we determine the invariant trace field and quaternion algebra of the Weeks manifold, the closed hyperbolic 3-manifold of smallest volume, and use this knowledge to prove that it contains no immersed totally geodesic surfaces.

In Section 5 the reader is introduced to orders in quaternion algebras defined over number fields and p-adic fields. The goal is to develop the theory needed to define arithmetic hyperbolic 3-manifolds, the subject of Section 6.

Section 6 defines arithmetic hyperbolic 3-manifolds and Kleinian groups. In fact they are defined in three different ways. The first definition employs a construction that naturally generalizes the construction of the Fuchsian group $\mathrm{SL}_2(\mathbb{Z})$ from the maximal order $\mathrm{M}_2(\mathbb{Z})$ in the quaternion algebra $\mathrm{M}_2(\mathbb{Q})$. Owing to the explicitness of this construction, we are able to easily show that certain aspects of the topology of an arithmetic hyperbolic 3-manifold can be deduced from the structure of the associated quaternion algebra; e.g., compactness. Our second characterization of arithmeticity shows that a hyperbolic 3-manifold is arithmetic precisely when its invariant trace field and quaternion algebra satisfy certain arithmetic properties. This characterization is then used to show that the Weeks manifold is arithmetic. Our final characterization is due to Margulis and shows that a finite volume hyperbolic 3-manifold is arithmetic precisely when it has infinite index inside of its commensurator.

Having defined arithmetic hyperbolic 3-manifolds and their number theoretic invariants, the notes now focus on applications to spectral geometry. In Section 7 we prove that compact arithmetic hyperbolic 3-manifolds which are isospectral are necessarily commensurable, a result due to Reid [31].

Section 8 is in many ways the highlight of the notes and gives a (somewhat modernized) exposition of Vignéras' construction of isospectral arithmetic hyperbolic 3-manifolds [35]. In our exposition this construction is based upon a theorem of Chinburg and Friedman concerning the embedding theory of maximal orders in quaternion algebras defined over number fields. Although the proof of this result may be taken as a black box, it is given in Section 9.

The remainder of the notes is devoted to applications of Vignéras' construction. In Section 10 we use Borel's formula for the volume of an arithmetic hyperbolic 3-manifold to show that Vignéras' method never produces more than cV^2 isospectral non-isometric hyperbolic 3-manifolds, where $c > 0$ is a constant and V is the volume of the manifolds in question. Section 11 recalls Sunada's construction of isospectral Riemannian manifolds. This method reduces the problem of constructing isospectral manifolds to a problem in finite group theory and is responsible for the vast majority of known examples of isospectral non-isometric Riemannian manifolds (though there are several notable exceptions). The main result of this section is that Vignéras' method is incompatible with Sunada's method in the sense that the latter cannot be used to construct Vignéras' examples.

The Magma [5] computer algebra system provides a large number of tools for studying quaternion algebras defined over fields of arithmetic interest. Throughout these notes we provide the code needed for the reader to calculate the quaternion algebras of certain explicitly defined hyperbolic 3-manifolds and covolumes of arithmetic Kleinian groups such as the Bianchi groups.

Much of the exposition in these notes, in particular in the early chapters where invariant trace fields, invariant quaternion algebras and arithmetic hyperbolic 3-manifolds are first defined, was taken from the excellent text of Maclachlan and Reid, *The arithmetic of hyperbolic* 3*-manifolds* [24]. Indeed, in many ways these notes were written to serve as a more streamlined version of [24] for readers with a particular interest in applications to spectral geometry.

2. Hyperbolic manifolds

As was mentioned in the introduction, these notes are intended for an audience familiar with the geometry of hyperbolic 3-manifolds and Kleinian groups and is meant to introduce *arithmetic* hyperbolic 3-manifolds and explore some of the distinguishing features of their geometry. In this section we will provide a rapid introduction to hyperbolic 3-manifolds and Kleinian groups. Our exposition is based on [24, Chapter 1] and [32]. As the reader is expected to already be familiar with much of this material, we will often omit proofs. There are a number of wonderful book length references which the reader interested in a more detailed treatment may consult. See for instance [3], [25] and [26].

2.1. **Hyperbolic 3-space.**

We begin by defining hyperbolic 3-space, which will always be regarded in the upper-half space model

$$\mathbf{H}^3 = \{(z,t) : z \in \mathbb{C}, t > 0\}$$

and as being equipped with the metric

$$ds^2 = \frac{|dz|^2 + dt}{t^2}.$$

In this manner $\mathbf{H}^3$ is the unique connected, simply-connected 3-dimensional Riemannian manifold with constant sectional curvature -1. We will view the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ as the *sphere at infinity* corresponding to $t = 0$. Geodesics in $\mathbf{H}^3$ are vertical Euclidean lines or semicircles orthogonal to $\hat{\mathbb{C}}$.

2.2. **Kleinian groups.**

A *Kleinian group* is a discrete subgroup of orientation preserving isometries of hyperbolic 3-space $\mathbf{H}^3$. It was already known to Poincaré that the group $\mathrm{Isom}^+(\mathbf{H}^3)$ of all orientation preserving isometries of hyperbolic 3-space is isomorphic to $\mathrm{PSL}_2(\mathbb{C})$, hence a Kleinian group is simply a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$.

While there are many excellent reasons to study Kleinian groups, our interest in them is due primarily to the following:

Theorem 2.1. *If M is an orientable hyperbolic 3-manifold then M is isometric to $\mathbf{H}^3/\Gamma$ where Γ is a torsion-free Kleinian group.*

The elements of $\mathrm{PSL}_2(\mathbb{C})$ induce biholomorphic maps of $\hat{\mathbb{C}}$ given by fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \left(z \mapsto \frac{az+b}{cz+d} \right).$$

These fractional linear transformations of $\hat{\mathbb{C}}$ extend to maps of $\mathbf{H}^3$ via the Poincaré extension. The Poincaré extension can be described geometrically as follows. Every fractional linear transformation of $\hat{\mathbb{C}}$ may be decomposed into a composition of inversions in circles and lines of $\hat{\mathbb{C}}$. Given such a circle or line, there is a unique hemisphere or plane in $\mathbf{H}^3$ which is orthogonal to $\hat{\mathbb{C}}$ and meets $\hat{\mathbb{C}}$ precisely at that circle or line. The Poincaré extension is simply the corresponding composition of inversions in $\mathbf{H}^3$. More concretely, the extension is given by the formula:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \left((z,t) \mapsto \left(\frac{(az+b)\overline{(cz+d)} + a\bar{c}t^2}{|cz+d|^2 + |c|^2t^2}, \frac{t}{|cz+d|^2+|c|^2t^2} \right) \right).$$

For example, the translation $z \mapsto z+1$ extends to the map $(z,t) \mapsto (z+1,t)$.

2.3. **Classification of isometries.** Let γ be a non-identity element of $\mathrm{PSL}_2(\mathbb{C})$. By examining the potential Jordan Normal Forms of γ, we see that γ must be conjugate to the image in $\mathrm{PSL}_2(\mathbb{C})$ of one of the following **class representatives**:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

We will refer to the isometry of $\hat{\mathbb{C}}$ induced by the class representative of γ as a **canonical form**. Consider first the class representative $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Its canonical form is the translation $z \mapsto z+1$. The canonical form of the class representative $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, on the other hand, is $z \mapsto \kappa z$ where $\kappa = \lambda^2$ is the isometry's **multiplier**.

Observe that if z is the fixed point of an isometry γ and $\gamma' \in \mathrm{PSL}_2(\mathbb{C})$ then $\gamma'(z)$ is the fixed point of $\gamma'\gamma\gamma'^{-1}$. From this we may conclude that a non-trivial element γ of $\mathrm{PSL}_2(\mathbb{C})$ has one fixed point in $\hat{\mathbb{C}}$ if its class representative is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and two fixed points otherwise. In the case that γ has two fixed points there is always a unique geodesic in $\mathbf{H}^3$ joining the points. We call this geodesic the **axis** of γ and denote it by A_γ. From this discussion we may conclude the following.

Lemma 2.2. *If the isometry associated to an element $\gamma \in \mathrm{PSL}_2(\mathbb{C})$ has at least 3 fixed points then $\gamma = \mathrm{Id}$.*

We are now ready to give the classification of isometries in $\gamma \in \mathrm{PSL}_2(\mathbb{C})$. We will do so in terms of the trace of γ, which is invariant under conjugation. Because we are working within $\mathrm{PSL}_2(\mathbb{C})$, where traces are only defined up to a sign, it is convenient to state our classification in terms of $\mathrm{tr}^2\,\gamma$.

- γ is **elliptic** if $\mathrm{tr}^2\,\gamma \in \mathbb{R}$ and $\mathrm{tr}^2\,\gamma < 4$.
- γ is **parabolic** if $\mathrm{tr}^2\,\gamma = 4$.
- γ is **loxodromic** if $\mathrm{tr}^2\,\gamma \in \mathbb{C} \setminus [0,4]$.
- γ is **hyperbolic** if $\mathrm{tr}^2\,\gamma \in \mathbb{R}$ and γ is loxodromic.

When γ is elliptic its multiplier κ satisfies $|\kappa| = 1$. When γ is hyperbolic $\kappa \in \mathbb{R}_{>0}$ and γ is loxodromic precisely when $|\lambda| \neq 1$. Observe that this already implies that whenever γ induces a finite order isometry it is necessarily the case that γ is elliptic.

Example 2.3. The above discussion allows us to give our first examples of Kleinian groups: the cyclic groups $\langle \gamma^n : n \in \mathbb{Z} \rangle$ where $\gamma \in \mathrm{PSL}_2(\mathbb{C})$. These groups are infinite unless γ is elliptic of finite order. Note as well that every element of these groups has the same set of fixed points in $\hat{\mathbb{C}}$.

These examples motivate the following easy lemma.

Lemma 2.4. *Let Γ be a Kleinian group with $\gamma_1, \gamma_2 \in \Gamma$. Then γ_1, γ_2 either have all of their fixed points in common or none at all.*

Proof. Clearly we may assume that at least one of γ_1, γ_2 is not parabolic, as the lemma is vacuously true otherwise. Suppose that γ_1, γ_2 have a single fixed point in common and normalize so that this fixed point is ∞. The lemma now follows from the observation that the sequence of elements given by $\gamma_1^{-n}\gamma_2\gamma_1^n$ contradicts the discreteness of Γ. As an example of how this argument is carried suppose that γ_1 is loxodromic and γ_2 is parabolic. By normalizing and possibly replacing γ_1 by γ_1^{-1}, it suffices to assume that γ_2 fixes ∞, that the two fixed points of γ_1 are $\{0, \infty\}$ and that

$$\gamma_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad |\lambda| > 1, \qquad \gamma_2 = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}, \qquad c \in \mathbb{C}.$$

We now compute that

$$\gamma_1^{-n}\gamma_2\gamma_1^n = \begin{pmatrix} 1 & c\lambda^{-2n} \\ 0 & 1 \end{pmatrix}.$$

Therefore $\gamma_1^{-n}\gamma_2\gamma_1^n \longrightarrow \mathrm{Id}$, which contradicts the discreteness of Γ. □

Lemma 2.4 allows us to deduce that if two loxodromic isometries do not have disjoint axes then their axes in fact coincide, and that a loxodromic element can never share a fixed point with a parabolic element.

Remark 2.5. *A more interesting example of a Kleinian group would be the group* $\mathrm{PSL}_2(\mathbb{Z}[i])$*, or more generally a* **Bianchi group** $\mathrm{PSL}_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-d})})$ *where d is a positive square-free integer. These groups will arise naturally when we construct Kleinian groups from orders in quaternion algebras, as they correspond to the case in which the quaternion algebra being considered is* $\mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$*.*

Definition 2.6. *Let Γ be a subgroup of* $\mathrm{PSL}_2(\mathbb{C})$*.*

- *The group Γ is* **reducible** *if all elements of γ have a common fixed point in their action on $\hat{\mathbb{C}}$. Otherwise Γ is* **irreducible***.*
- *The group Γ is* **elementary** *if it is virtually abelian; that is, it contains an abelian subgroup of finite index.*

Remark 2.7. *An equivalent definition of elementary is that any two elements of infinite order have a common fixed point.*

All of the groups in Example 2.3 are cyclic and therefore abelian, hence elementary. The classification of torsion-free elementary Kleinian groups turns out not to be much more complicated.

Theorem 2.8. *If Γ is a torsion-free elementary Kleinian group then Γ is one of the following abelian groups:*

(1) $\langle \gamma^n : n \in \mathbb{Z} \rangle$ where γ is parabolic
(2) $\langle \gamma^n : n \in \mathbb{Z} \rangle$ where γ is loxodromic
(3) $\langle \gamma_1^n\gamma_2^m : n, m \in \mathbb{Z} \rangle$ where γ_1, γ_2 are parabolic elements with a common fixed point but different translation directions

In fact the classification of *all* elementary Kleinian groups is known (see [3]).

In contrast to the examples given above, non-elementary Kleinian groups can be quite complicated, as the following theorem makes clear.

Theorem 2.9. *Every non-elementary Kleinian group contains infinitely many loxodromic elements, no two of which have a common fixed point.*

Given a subgroup of $\mathrm{PSL}_2(\mathbb{C})$, it can be challenging to decide whether the group is discrete. Two of the best results that are known in this direction are due to Jørgensen [19, 20].

Theorem 2.10 (Jørgensen)**.** *Let $x, y \in \mathrm{PSL}_2(\mathbb{C})$ and $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ be a non-elementary subgroup.*

(1) Γ is discrete if and only if every two-generator subgroup of Γ is discrete.
(2) $\langle x, y\rangle$ is discrete if and only if

$$|\operatorname{tr}^2 x - 4| + |\operatorname{tr}[x,y] - 2| \geq 1,$$

where $[x,y] = xyx^{-1}y^{-1}$ is the commutator of x and y.

As an immediate consequence of Jørgensen's inequality we have an interesting lemma of Shimizu [33, Lemma 4].

Lemma 2.11 (Shimizu)**.** *If Γ is a non-elementary Kleinian group containing a parabolic element $x = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$ then every element $y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ satisfies $|c\mu| \geq 1$.*

Proof. As $\operatorname{tr}[x,y] = \operatorname{tr}(xyx^{-1}y^{-1}) = 2 + c^2\mu^2$, the lemma is a direct consequence of Theorem 2.10(2). □

We should note that Shimizu's original proof of Lemma 2.11, like Jørgensen's proof of Theorem 2.10(2), relies heavily on elementary matrix manipulations.

Irreducible two-generator groups can be easily identified by means of the following linear independence condition.

Proposition 2.12. *Let $x, y \in \mathrm{PSL}_2(\mathbb{C})$. Then $\langle x, y\rangle$ is irreducible if and only if the vectors $1, x, y, xy$ are linearly independent in $\mathrm{M}_2(\mathbb{C})$.*

2.4. **Geodesics.** Let Γ be a Kleinian group with finite covolume and $\gamma \in \Gamma$ be a non-identity element which is not parabolic. Then Γ has two fixed points in $\hat{\mathbb{C}}$ and the axis A_γ of Γ is the unique geodesic in $\mathbf{H}^3$ joining these fixed points. In terms of its induced isometry of $\mathbf{H}^3$, γ moves points along A_γ by some distance $\ell_0(\gamma)$ and rotates them about A_γ by an angle $\theta(\gamma)$. We call $\ell_0(\gamma)$ the **translation length** of γ and $\theta(\gamma)$ the **rotation angle** of γ. The quantity

$$\ell(\gamma) = \ell_0(\gamma) + i\theta(\gamma)$$

is called the **complex length** of γ. These quantities are all connected by virtue of the formula

$$\operatorname{tr}(\gamma) = 2\cosh\left(\frac{\ell(\gamma)}{2}\right). \tag{1}$$

Suppose now that γ is loxodromic and let $\lambda(\gamma)$ denote its eigenvalue satisfying $|\lambda(\gamma)| > 1$. Then we have the following useful formula for the translation length of γ:

$$\ell_0(\gamma) = 2\log|\lambda(\gamma)|.$$

A few observations about the translation length are in order. When γ is hyperbolic it is always that case that $\theta(\gamma) = 0$. When γ is elliptic on the other hand, $\ell_0(\gamma) = 0$ and γ simply rotates points about its axis. Note that in this case γ will have finite order precisely when $\theta(\gamma) \in 2\pi\mathbb{Q}$. Finally, by convention we have that $\ell_0(\gamma) = \theta(\gamma) = 0$ when γ is parabolic.

Now let $M = \mathbf{H}^3/\Gamma$ be a complete orientable hyperbolic 3-manifold of finite covolume. Given a loxodromic element $\gamma \in \Gamma$, the axis A_γ projects to a closed geodesic on M. Moreover, the length of this closed geodesic on M is the translation length $\ell_0(\gamma)$. Now consider a homotopically non-trivial closed loop Q in M. Because M is a closed Riemannian manifold of negative curvature, Q is freely homotopic to a unique closed geodesic. We now define the length of Q to be the translation length of the unique (up to conjugacy) loxodromic element of Γ whose axis projects onto Q. We therefore have a correspondence between closed geodesics on M and (conjugacy classes of) loxodromic elements in $\pi_1(M)$. These geodesic lengths may thus be studied by means of the trace of the corresponding loxodromic element via equation (1). An important observation is that, modulo a slight ambiguity arising from the fact that $\operatorname{tr}\gamma$ is only defined up to ± 1, the number of closed geodesics on M having a fixed length corresponds to the number of conjugacy classes of elements of Γ having a certain fixed minimal polynomial.

We conclude our discussion by recording an important consequence of Theorem 2.9.

Theorem 2.13. *If Γ is a non-elementary Klienian group then $M = \mathbf{H}^3/\Gamma$ contains infinitely many distinct closed geodesics.*

2.5. **Commensurability.** We now give the definition of *commensurability*, a notion that will underlie much of what is to follow in these notes. Indeed, one of our main goals will be to attach to a finite volume hyperbolic 3-manifold invariants which depend only on the manifold's commensurability class. The notion of commensurability is quite natural from the standpoint of arithmetic hyperbolic 3-manifolds, as we will see that the commensurability class of such a manifold corresponds to a certain *quaternion algebra*, a number theoretic object for which there is a rich structure theory.

Definition 2.14. *Let Γ_1, Γ_2 be subgroups of $\mathrm{PSL}_2(\mathbb{C})$.*

- *We say that Γ_1 and Γ_2 are* **directly commensurable** *if $\Gamma_1 \cap \Gamma_2$ has finite index in both Γ_1 and Γ_2. We say that Γ_1 and Γ_2 are* **commensurable in the wide sense** *if Γ_1 and a conjugate of Γ_2 are directly commensurable.*
- *Let M_1, M_2 be hyperbolic 3-manifolds (or orbifolds). We say that M_1 and M_2 are* **commensurable** *if they have a common finite sheeted hyperbolic cover.*

Note that in the definition of commensurable, the common cover will usually only be considered up to isometry. In this case the two manifolds / orbifolds will be commensurable if and only if their fundamental groups are commensurable in the wide sense. It is for this reason that we will be interested primarily in commensurability in the wide sense.

Within a commensurability class we can also pass from orbifolds to manifolds (or equivalently, from Kleinian groups to torsion-free Kleinian groups) by virtue of Selberg's Lemma.

Theorem 2.15 (Selberg's Lemma)**.** *If Γ is a finitely generated subgroup of $\mathrm{GL}_n(\mathbb{C})$ then Γ contains a torsion-free subgroup of finite index.*

3. Quaternion algebras

One of the themes throughout these notes is the idea that much of the geometry of hyperbolic 3-manifolds can be characterized in an algebraic manner and studied using powerful techniques from non-commutative algebra and number theory. Crucial to this characterization is the notion of a quaternion algebra. We will see that to every Kleinian group of finite covolume is a quaternion algebra defined over a number field. In this section we will review some of the basic properties of quaternion algebras.

Unless explicitly stated otherwise, throughout this section we will denote by R a commutative ring and by k a field of characteristic other than 2.

3.1. **Central simple algebras: Generalities.** We begin with a few definitions.

Definition 3.1. *A ring A is an R-**algebra** if A has the structure of an R-module in which the ring and module operations satisfy the compatibility condition $r(ab) = (ra)b = a(rb)$ for all $r \in R$ and $a, b \in A$.*

Example 3.2. Every ring A has the structure of a $\mathbb{Z}$-algebra. This structure is induced by the unique homomorphism $\mathbb{Z} \hookrightarrow A$ mapping 1 to the multiplicative identity of A.

Example 3.3. Let X be a topological space and $C(X, \mathbb{R})$ be the set of continuous real-valued functions $f : X \to \mathbb{R}$. The set $C(X, \mathbb{R})$ has the structure of a commutative ring where functions are added and multiplied pointwise (i.e., $(f+g)(x) = f(x)+g(x)$ and $(fg)(x) = f(x)g(x)$). In fact it is not hard to see that $C(X, \mathbb{R})$ is an $\mathbb{R}$-algebra via the constant functions $r \in \mathbb{R} \mapsto f(x) \equiv r$.

Example 3.4. Let k be a field, V a k-vector space and $\mathrm{End}_k(V)$ the set of k-linear transformations from V to itself. The set $\mathrm{End}_k(V)$ is a ring (called the *endomorphism ring of* V) where addition is defined pointwise and multiplication is given by composition of functions. Note that $\mathrm{End}_k(V)$ has the structure of a k-algebra via the map $k \hookrightarrow \mathrm{End}_k(V)$ given by $a \mapsto (v \mapsto av)$. When $V = k^n$ the resulting endomorphism algebra is isomorphic to the matrix algebra $\mathrm{M}_n(k)$.

In what follows A is always assumed to be an R-algebra.

Definition 3.5. *The algebra A is **simple** if it is simple as an R-module. That is, A contains no proper nontrivial submodules. Equivalently, A contains no proper nontrivial two-sided ideals.*

Let $Z(A)$ denote the center of A when regarded as a ring. That is,

$$Z(A) = \{a \in A : ab = ba \text{ for all } b \in A\}.$$

By definition it is always the case that $R \subseteq Z(A)$.

Definition 3.6. *The algebra A is a **division algebra** if every non-zero element of A has a multiplicative inverse.*

Definition 3.7. *The algebra A is **central** if $R = Z(A)$.*

Definition 3.8. *The **dimension** of a k-algebra A is the dimension of A as a k-vector space.*

We now define this section's main object of study.

Definition 3.9. *A* **central simple algebra** *(over k) is a k-algebra which is both central and simple.*

Example 3.10. If D is k-algebra which is a division algebra then D is simple. When D is central as well we will refer to it as a **central division algebra**.

Example 3.11. Let L/k be a degree n extension of fields so that L has the structure of a k-algebra. As a k-algebra L has dimension n and is simple, but is not central.

Example 3.12. We now return to the example $C(X, \mathbb{R})$ defined above. This algebra is neither central nor simple. The former assertion is an immediate consequence of $C(X, \mathbb{R})$ being commutative, while the latter assertion follows from the existence of the two-sided ideal consisting of all functions $f \in C(X, \mathbb{R})$ such that $f(x_0) = 0$ (where $x_0 \in X$ is any fixed element). In general the algebra $C(X, \mathbb{R})$ will not be finite dimensional. When $X = \mathbb{R}$ for instance, an infinite linearly independent set is given by the polynomial functions $\{1, x, x^2, x^3, \dots\}$.

Example 3.13. Our first example of a central simple algebra is the algebra $\mathrm{M}_n(k)$. The center of $\mathrm{M}_n(k)$ consists of all scalar multiples of the matrix $\mathrm{diag}(1, \dots, 1)$, and an easy argument with elementary matrices implies that $\mathrm{M}_n(k)$ has no two-sided ideals. The dimension of $\mathrm{M}_n(k)$ is n^2.

Example 3.14. Our second example of a central simple algebra is the four dimensional division algebra $\mathbb{H}$ of Hamilton's quaternions. This is the $\mathbb{R}$-algebra with basis $\{1, i, j, ij\}$ subject to the relations

$$i^2 = j^2 = -1$$

and $ij = -ji$. We will later see that $\mathbb{H}$ is the *unique* four dimensional central division algebra over $\mathbb{R}$.

The following is a fundamental theorem in the study of central simple algebras.

Theorem 3.15 (Skolem-Noether)**.** *Let k be a field, A be a finite-dimensional central simple algebra over k and B be a finite-dimensional simple k-algebra. If $f_1, f_2 : B \longrightarrow A$ are algebra homomorphisms then there exists an element $a \in A^*$ such that $f_2(b) = a^{-1} f_1(b) a$ for all $b \in B$.*

The following corollary of Theorem 3.15 is important enough that it is also often referred to as the Skolem-Noether Theorem.

Corollary 3.16. *Every automorphism of a central simple algebra is an inner automorphism.*

Proof. This follows from the Skolem-Noether theorem upon taking $A = B$, f_1 the identity map and f_2 an arbitrary automorphism of A. □

Crucial to the theory of central simple algebras is the structure theorem of Wedderburn which characterizes central simple algebras as matrix algebras taking coefficients in a central division algebra.

Theorem 3.17 (Wedderburn)**.** *Let A be a central simple algebra over a field k. Then there is a central division algebra D over k such that $A \cong \mathrm{M}_n(D)$ for some $n \geq 1$.*

3.2. **Quaternion algebras: Generalities.**

Definition 3.18. *A* **quaternion algebra** *over k is a four-dimensional central simple algebra over k with basis $\{1, i, j, ij\}$ satisfying the relations*

$$i^2 = a, \qquad j^2 = b, \qquad ij = -ji$$

for some $a, b \in k^*$.

We will denote the quaternion algebra in Definition 3.18 via its *Hilbert symbol* $\left(\frac{a,b}{k}\right)$. Also note that the terminology "quaternion algebra" is motivated by the fact that the algebras defined above generalize Hamilton's construction of $\mathbb{H}$, which in our notation corresponds to the algebra $\left(\frac{-1,-1}{\mathbb{R}}\right)$.

An important property of quaternion algebras is that they behave very nicely with respect to extension of scalars. That is, if k' is a field containing k then we have

$$\left(\frac{a,b}{k}\right) \otimes_k k' \cong \left(\frac{a,b}{k'}\right).$$

To make the utility of this property more apparent we will prove that the quaternion algebra $\left(\frac{a,b}{k}\right)$ is always a central simple algebra of dimension 4.

Proposition 3.19. *The quaternion algebra $\left(\frac{a,b}{k}\right)$ is a four dimensional central simple algebra.*

Proof. That $\left(\frac{a,b}{k}\right)$ is four dimensional as a k-algebra is clear. We first show that $\left(\frac{a,b}{k}\right)$ is central. Let $\widehat{k}$ be an algebraic closure of k and consider the $\widehat{k}$-algebra

$$\left(\frac{a,b}{k}\right) \otimes_k \widehat{k} \cong \left(\frac{a,b}{\widehat{k}}\right).$$

By Wedderburn's theorem this algebra is either a central division algebra or else is isomorphic to $\mathrm{M}_2(\widehat{k})$. To see that it cannot be a central division algebra, we argue as follows. Let $z \in \left(\frac{a,b}{\widehat{k}}\right)$ be a non-scalar element and consider the set of powers $\{z^n : n \in \mathbb{Z}_{\geq 0}\}$. This set cannot be linearly independent because $\left(\frac{a,b}{\widehat{k}}\right)$ has finite dimension, hence there is an irreducible polynomial $f \in \widehat{k}[x]$ of least degree such that $f(z) = 0$. But since $\widehat{k}$ is algebraically closed we must have $z \in \widehat{k}$, a contradiction. It follows that $\left(\frac{a,b}{\widehat{k}}\right) \cong \mathrm{M}_2(\widehat{k})$. The latter algebra is known to be central with center $\widehat{k}$, hence $\left(\frac{a,b}{k}\right)$ is central as well.

Our proof that $\left(\frac{a,b}{k}\right)$ is simple is similar. Let I be a non-zero two-sided ideal of $\left(\frac{a,b}{k}\right)$ and consider the ideal $I \otimes_k \widehat{k}$ of $\mathrm{M}_2(\widehat{k})$. Because the latter is simple, $I \otimes_k \widehat{k}$ is four dimensional as a vector space over $\widehat{k}$. It follows that I is four dimensional as a vector space over k, hence $I = \left(\frac{a,b}{k}\right)$. □

We will now show that not only are quaternion algebras examples of four dimensional centra simple algebras, but in fact every four dimensional centra simple algebra is a quaternion algebra.

Theorem 3.20. *Let A be a four dimensional central simple algebra over k. There exist $a, b \in k^*$ such that $A \cong \left(\frac{a,b}{k}\right)$.*

Proof. Wedderburn's theorem (Theorem 3.17) implies that if A is not a division algebra then $A \cong \mathrm{M}_2(k)$. In the latter case $\left(\frac{1,1}{k}\right) \cong A$ via the isomorphism induced by

$$i \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$j \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We may therefore assume that A is a division algebra. Let $x \in A \setminus k$ and consider the quadratic field extension $L = k(x)/k$. Because k has characteristic not equal to 2 there exists an element $i \in L$ such that $i^2 \in k$ and $L = k(i)$. Denote by σ the non-trivial automorphism of L/k; that is, the map induced by $\sigma(i) = -i$. By the Skolem-Noether theorem there exists an element $j \in A^*$ such that

$$jij^{-1} = \sigma(i) = -i.$$

In other words, j satisfies $ij = -ji$. Clearly the element j does not lie in $L = k(i)$. In fact we claim that $\{1, i, j, ij\}$ is a basis for A. To see this, suppose that $ij = \alpha + \beta i + \gamma j$ with $\alpha, \beta, \gamma \in k$ and note that the equality

$$j = \frac{\alpha + \beta i}{i - \gamma}$$

implies that $j \in L = k(i)$, a contradiction. We will now show that $j^2 \in k$. It suffices to show that j^2 lies in the center of A. To that end we must show that j^2 commutes with i. But this follows from the relations $jij^{-1} = -i$ and $ij = -ji$ (which imply that $j^2ij^{-2} = i$). This concludes the proof as we have shown that $A \cong \left(\frac{a,b}{k}\right)$ where $a = i^2$ and $b = j^2$. □

We note that while Theorem 3.20 guarantees that if A is a four dimensional central simple algebra over k then there exist $a, b \in k^*$ such that $A \cong \left(\frac{a,b}{k}\right)$, it is not the case that a, b uniquely determine the isomorphism class of A. This is illustrated in the following result, which shows that the isomorphism class of $\left(\frac{a,b}{k}\right)$ is unchanged upon multiplying a or b by squares.

Proposition 3.21. *If $a, b, x, y \in k^*$ then*

$$\left(\frac{a, b}{k}\right) \cong \left(\frac{ax^2, by^2}{k}\right).$$

Proof. Let $\{1, i, j, ij\}$ and $\{1, i', j', i'j'\}$ be bases for $\left(\frac{a,b}{k}\right)$ and $\left(\frac{ax^2,by^2}{k}\right)$ and

$$\phi : \left(\frac{ax^2, by^2}{k}\right) \to \left(\frac{a, b}{k}\right)$$

be the homomorphism obtained by defining $\phi(1) = 1$, $\phi(i') = xi$, $\phi(j') = yj$, $\phi(i'j') = xyij$ and extending linearly. The image of ϕ is the k-subalgebra of $\left(\frac{a,b}{k}\right)$ with basis $\{1, xi, yj, xyij\}$. As this subalgebra has dimension four over k, it must coincide with $\left(\frac{a,b}{k}\right)$. In other words, ϕ is surjective. Any surjective homomorphism between k-algebras of the same dimension is an isomorphism, so the proposition follows. □

It is, of course, useful to know when $\left(\frac{a,b}{k}\right)$ is isomorphic to $\mathrm{M}_2(k)$ and when it is a division algebra (which by Wedderburn's theorem are the only two possibilities). The following proposition will be very useful in this regard.

Proposition 3.22. *The quaternion algebras* $\left(\frac{1,b}{k}\right)$ *and* $\mathrm{M}_2(k)$ *are isomorphic for any* $b \in k^*$.

Proof. The desired isomorphism is given by the map

$$\psi : \left(\frac{1,b}{k}\right) \longrightarrow \mathrm{M}_2(k),$$

where

$$\psi(x + yi + zj + wij) = \begin{pmatrix} x+y & z+w \\ b(z-t) & x-y \end{pmatrix}.$$

The inverse is the map defined by

$$\psi^{-1}\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) = \frac{1}{2}(\alpha + \delta + (\alpha - \delta)i + (\beta + b^{-1}\gamma)j + (\beta - b^{-1}\gamma)ij).$$

□

We conclude this section by proving an important property about subfields of quaternion algebras.

Proposition 3.23. *Let A be a quaternion algebra over a field k and L/k a quadratic field extension. If there exists an embedding of k-algebras $L \hookrightarrow A$ then $A \otimes_k L \cong \mathrm{M}_2(L)$.*

Proof. By identifying L with its image in A it suffices to assume that $L \subset A$. We may select a basis $\{1, i, j, ij\}$ of A such that $L = k(i)$ and $i^2 = a \in k^*$. We now have

$$A \otimes_k L \supset L \otimes_k L \cong L \otimes_k k[x]/(x^2 - a) \cong L[x]/(x^2 - a) \cong k \oplus k.$$

The algebra $k \oplus k$ contains zero divisors, hence $A \otimes_k L$ is not a division algebra. By Theorem 3.17 we have that $A \otimes_k L \cong \mathrm{M}_2(L)$. □

3.3. **Quaternion algebras over the complex numbers.** Quaternion algebras over $\mathbb{C}$ are very easy to understand. There is a unique such quaternion algebra: $\mathrm{M}_2(\mathbb{C})$.

Theorem 3.24. *If A is a quaternion algebra over $\mathbb{C}$ then $A \cong \mathrm{M}_2(\mathbb{C})$.*

Proof. The fundamental theorem of algebra implies that every element of $\mathbb{C}^*$ is a square, hence $A \cong \left(\frac{1,1}{\mathbb{C}}\right)$ by Proposition 3.21. The latter quaternion algebra is isomorphic to $\mathrm{M}_2(\mathbb{C})$ by Proposition 3.22. □

3.4. **Quaternion algebras over the real numbers.** The structure of quaternion algebras over $\mathbb{R}$ is more complicated than over $\mathbb{C}$, but only just. Up to isomorphism there are only two quaternion algebras over $\mathbb{R}$: $\mathrm{M}_2(\mathbb{R})$ and $\mathbb{H}$.

Theorem 3.25. *If A is a quaternion algebra over $\mathbb{R}$ then $A \cong \mathrm{M}_2(\mathbb{R})$ or $A \cong \mathbb{H}$.*

Proof. Proposition 3.21 implies that A is isomorphic to one of the following three quaternion algebras: $\left(\frac{-1,-1}{\mathbb{R}}\right)$, $\left(\frac{1,-1}{\mathbb{R}}\right)$ or $\left(\frac{1,1}{\mathbb{R}}\right)$. The first of these algebras is isomorphic to $\mathbb{H}$ by definition, while the second and third are isomorphic to $\mathrm{M}_2(\mathbb{R})$ by Proposition 3.22. □

3.5. **Quaternion algebras over p-adic fields.** Let k be a p-adic field with fixed uniformizer π. As was the case for $\mathbb{R}$, there are precisely two isomorphism classes of quaternion algebras over k. Moreover, we once again have an explicit description of the unique quaternion division algebra over k. As the proof would take us too far afield, we will simply state the following result and refer the reader to [36] for a more details.

Theorem 3.26. *The k-algebra $\left(\frac{u,\pi}{k}\right)$ is the unique quaternion division algebra over k, where $k(\sqrt{u})$ is the unique unramified quadratic extension of k.*

3.6. **Quaternion algebras over number fields.** Let k be a number field, $a, b \in k^*$ and consider the quaternion algebra $\left(\frac{a,b}{k}\right)$. If K is a field containing k then we may obtain a K-quaternion algebra from $\left(\frac{a,b}{k}\right)$ via extension of scalars: $\left(\frac{a,b}{k}\right) \otimes_k K \cong \left(\frac{a,b}{K}\right)$. In the study of the structure of quaternion algebras over number fields one often chooses K to be a completion of k (i.e., $\mathbb{C}, \mathbb{R}$ or a p-adic field $k_{\mathfrak{p}}$ for some prime $\mathfrak{p}$ of k) and studies the algebra over K obtained by extension of scalars. The hope, of course, is that one can then deduce information about the structure of the original algebra over k.

To make all of this more precise, let $\{1, i, j, ij\}$ be the standard basis for $\left(\frac{a,b}{k}\right)$ and $\sigma : k \hookrightarrow K$ a fixed embedding.

Lemma 3.27. *There is an isomorphism*

$$\left(\frac{a,b}{k}\right) \otimes_\sigma K \cong \left(\frac{\sigma(a), \sigma(b)}{K}\right).$$

Proof. Let $\{1, i', j', i'j'\}$ be the standard basis for $\left(\frac{\sigma(a),\sigma(b)}{K}\right)$. The desired isomorphism is the one assigning

$$(a_0 + a_1 i + a_2 j + a_3 ij) \otimes_\sigma \alpha \mapsto \alpha(\sigma(a_0) + \sigma(a_1)i' + \sigma(a_2)j' + \sigma(a_3)i'j').$$

∎

As an example application of this, consider the quaternion algebra $\left(\frac{-1,-1}{\mathbb{Q}}\right)$. If $\sigma : \mathbb{Q} \to \mathbb{R}$ is the standard inclusion then Lemma 3.27 implies that $\left(\frac{-1,-1}{\mathbb{Q}}\right)$ is a division algebra (as $\sigma(-1) = -1$ and $\left(\frac{-1,-1}{\mathbb{R}}\right)$ is a division algebra). An interesting nuance in the theory of quaternion algebras over number fields is that unlike $\mathbb{R}$, there is not a unique quaternion division algebra over a number field. In fact, over every number field there are infinitely many isomorphism classes of quaternion algebras!

Definition 3.28. *Let k be a number field, v be a place of k with corresponding embedding σ, and k_v be the corresponding completion of k. We say that a quaternion algebra A over k is* **ramified at** σ **and** v *if $A \otimes_\sigma k_v$ is a division algebra. Otherwise we say that A is* **split at** σ **and** v.

Remark 3.29. *For convenience we will usually say that a quaternion algebra A over k is ramified at a place v of k and omit mention of the associated embedding σ.*

Remark 3.30. *Notice that if $A = \mathrm{M}_2(k)$ then $A \otimes_\sigma k_v \cong \mathrm{M}_2(k_v)$ for all σ, v. In particular every place of k is split in $\mathrm{M}_2(k)$.*

Recall that by Theorem 3.24, every quaternion algebra over k is split at all complex places. Thus only real or p-adic places may ramify.

Suppose now that k has r_1 real places and r_2 complex places. Denote by S_∞ the set of archimedean places of k. We therefore have isomorphisms

$$\begin{aligned} A \otimes_{\mathbb{Q}} \mathbb{R} &\cong \bigoplus_{v \in S_\infty} A \otimes_k k_v \\ &\cong \mathrm{M}_2(\mathbb{C})^{r_2} \times \bigoplus_{\sigma : k \hookrightarrow \mathbb{R}} A \otimes_\sigma k_v \\ &\cong \mathrm{M}_2(\mathbb{C})^{r_2} \times \mathrm{M}_2(\mathbb{R})^s \times \mathbb{H}^{r_1 - s}, \end{aligned}$$

where s is the number of real places of k at which A is split. In Section 6 we will see that arithmetic Kleinian groups are constructed from quaternion algebras in which $r_2 = 1$ and $s = 0$. A simple way of ensuring this is by taking k to be an imaginary quadratic field, in which case $s = 0$ since there are no real places.

Let $\mathrm{Ram}(A)$ denote the set of places of k (be they finite or infinite) at which A is ramified. The following theorem classifies quaternion algebras over number fields and implies, as was stated above, that there are finitely many isomorphism classes of quaternion division algebras over every number field. For a proof, see [36, Chapitre III.3].

Theorem 3.31 (Classification of Quaternion Algebras over Number Fields)**.** *Let k be a number field. If A is a quaternion algebra over k then* $\mathrm{Ram}(A)$ *is finite and of even cardinality. Conversely, given any finite set S of places of k (finite or infinite) with even cardinality there exists a unique quaternion algebra A over k such that* $\mathrm{Ram}(A) = S$.

The following is an immediate corollary of Theorem 3.31.

Corollary 3.32. *If k is a number field and A, A' are quaternion algebras over k then $A \cong A'$ if and only if* $\mathrm{Ram}(A) = \mathrm{Ram}(A')$.

4. Trace fields and quaternion algebras

In this section we will attach to a Kleinian group Γ of finite covolume two arithmetic objects: a number field and a quaternion algebra. These objects will turn out to be invariants of the commensurability class of Γ and will encode a great deal of the geometry of the hyperbolic 3-orbifold $\mathbf{H}^3/\Gamma$. For instance, we will use these invariants to show that the closed hyperbolic 3-manifold of smallest volume has no immersed totally geodesic surfaces.

4.1. Trace fields and quaternion algebras for finite covolume Kleinian groups.

Let Γ be a non-elementary subgroup of $\mathrm{PSL}_2(\mathbb{C})$ and $\hat{\Gamma} := P^{-1}(\Gamma)$ denote the preimage of Γ under the projection map $P : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{PSL}_2(\mathbb{C})$.

Definition 4.1. *The* **trace field** *of* Γ*, denoted* $\mathbb{Q}(\operatorname{tr}\Gamma)$*, is the field* $\mathbb{Q}(\operatorname{tr}\hat{\gamma} : \hat{\gamma} \in \hat{\Gamma})$*.*

The following result, proven in [24, Theorem 3.1.2], is a consequence of Mostow's Rigidity Theorem and an examination of the representation variety $\mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$.

Theorem 4.2. *If* Γ *is a Kleinian group of finite covolume then* $\mathbb{Q}(\operatorname{tr}\Gamma)$ *is a number field.*

We now associate to Γ a quaternion algebra defined over $\mathbb{Q}(\operatorname{tr}\Gamma)$. To begin with, define

$$A_0\Gamma = \left\{ \sum a_i\gamma_i : a_i \in \mathbb{Q}(\operatorname{tr}\Gamma), \gamma_i \in \Gamma \right\},$$

where only finitely many of the a_i are non-zero. We define multiplication in the obvious manner. For instance, $a_1\gamma_1 \cdot a_2\gamma_2 = a_1a_2(\gamma_1\gamma_2)$.

It is clear that $A_0\Gamma$ is an algebra over $\mathbb{Q}(\operatorname{tr}\Gamma)$. The following theorem shows that in fact, $A_0\Gamma$ is a quaternion algebra over $\mathbb{Q}(\operatorname{tr}\Gamma)$. During the course of the proof it will often be useful to consider the $\mathbb{C}$-algebra $A_0\Gamma \otimes_{\mathbb{Q}(\operatorname{tr}\Gamma)} \mathbb{C}$. To ease notation we will denote this algebra as $A_0\Gamma \otimes \mathbb{C}$. All other undecorated tensor products are with respect to $\mathbb{Q}(\operatorname{tr}\Gamma)$ as well.

Theorem 4.3. $A_0\Gamma$ *is a quaternion algebra over* $\mathbb{Q}(\operatorname{tr}\Gamma)$*.*

Proof. As we have already noted that $A_0\Gamma$ is a $\mathbb{Q}(\operatorname{tr}\Gamma)$-algebra, we must show that $A_0\Gamma$ is a four-dimensional central simple algebra over $\mathbb{Q}(\operatorname{tr}\Gamma)$.

Because Γ is non-elementary, it contains a pair of loxodromic elements g, h such that $\langle g, h\rangle$ is irreducible. This implies, by Proposition 2.12 that Id, g, h, gh are linearly independent in $\mathrm{M}_2(\mathbb{C})$. It follows that $A_0\Gamma \otimes \mathbb{C}$ is an algebra of dimension at least 4 over $\mathbb{C}$ which is contained in $\mathrm{M}_2(\mathbb{C})$, hence $A_0\Gamma \otimes \mathbb{C} = \mathrm{M}_2(\mathbb{C})$.

We now show that $A_0\Gamma$ has dimension exactly 4 over $\mathbb{Q}(\operatorname{tr}\Gamma)$. Let $T(\cdot,\cdot)$ denote the trace form on $\mathrm{M}_2(\mathbb{C})$; that is, the map $T(a, b) = \operatorname{tr}(ab)$. It is a non-degenerate symmetric bilinear form. Let $\{\mathrm{Id}^*, g^*, h^*, (gh)^*\}$ be a dual basis of $\mathrm{M}_2(\mathbb{C})$. Since this basis spans, for $\gamma \in \Gamma$ we have

$$\gamma = x_0\,\mathrm{Id}^* + x_1g^* + x_2h^* + x_3(gh)^*, \qquad x_i \in \mathbb{C}.$$

If $\gamma_i \in \{\mathrm{Id}, g, h, (gh)\}$ then $T(\gamma, \gamma_i) = \operatorname{tr}(\gamma\gamma_i) = x_j$ for some $j \in \{0, 1, 2, 3\}$. Since $\operatorname{tr}\gamma\gamma_i \in \mathbb{Q}(\operatorname{tr}\Gamma)$, we see that $x_1, \ldots, x_3 \in \mathbb{Q}(\operatorname{tr}\Gamma)$ as well. Thus

$$\mathbb{Q}(\operatorname{tr}\Gamma)[\mathrm{Id}, g, h, gh] \subset A_0\Gamma \subset \mathbb{Q}(\operatorname{tr}\Gamma)[\mathrm{Id}^*, g^*, h^*, (gh)^*].$$

Therefore $A_0\Gamma$ is four dimensional over $\mathbb{Q}(\operatorname{tr}\Gamma)$.

That $A_0\Gamma$ is central follows from the fact that $A_0\Gamma\otimes\mathbb{C}=\mathrm{M}_2(\mathbb{C})$ is central. Indeed, if $a\in A_0\Gamma$ is central, then a lies in the center of $A_0\Gamma\otimes\mathbb{C}=\mathrm{M}_2(\mathbb{C})$ as well. This implies that a is a multiple of the identity.

Finally, we must show that $A_0\Gamma$ is simple. Let I be a non-zero two-sided ideal of $A_0\Gamma$. Then $I\otimes\mathbb{C}$ is a non-zero two-sided ideal in $\mathrm{M}_2(\mathbb{C})$. As $\mathrm{M}_2(\mathbb{C})$ is simple, we must have $I\otimes\mathbb{C}=\mathrm{M}_2(\mathbb{C})$; that is, $I\otimes\mathbb{C}$ has dimension 4 over $\mathbb{C}$. Thus I must have dimension at least 4 over $\mathbb{Q}(\operatorname{tr}\Gamma)$, so a comparison of dimensions shows that $I=A_0\Gamma$. □

We remark that multiplication in $A_0\Gamma$ is simply the restriction of matrix multiplication in $\mathrm{M}_2(\mathbb{C})$. Similarly, the reduced trace and norm in $A_0\Gamma$ coincide with the usual matrix trace and determinant in $\mathrm{M}_2(\mathbb{C})$.

Corollary 4.4. *If Γ is a non-elementary Kleinian group and $g,h\in\Gamma$ are loxodromic elements for which $\langle g,h\rangle$ is irreducible, then $A_0\Gamma=\mathbb{Q}(\operatorname{tr}\Gamma)[\mathrm{Id},g,h,gh]$.*

In what follows we will let $k=\mathbb{Q}(\operatorname{tr}\Gamma)$.

Corollary 4.5. *Let Γ be a non-elementary Kleinian group with finite covolume and $\gamma\in\Gamma$ be a loxodromic element with eigenvalue λ. The group Γ is conjugate to a subgroup of $\mathrm{SL}_2(k(\lambda))$.*

Proof. Suppose first that for every loxodromic element $\gamma\in\Gamma$ the eigenvalue λ of γ is an element of k. In this case we have $k(\lambda)=k$, hence we must show that Γ is conjugate to a subgroup of $\mathrm{SL}_2(k)$. Let g,h be loxodromic elements of Γ with no common fixed points so that group $\langle g,h\rangle$, which is discrete by Theorem 2.10(1), is irreducible. Because of our assumption that the eigenvalues of g,h lie in k, we may conjugate so that all of the matrix entries of g and h lie in k. Corollary 4.4 shows that $A_0\Gamma=k[\mathrm{Id},g,h,gh]$, hence $A_0\Gamma\subset\mathrm{M}_2(k)$. It follows that upon conjugating, $\Gamma\subset\mathrm{SL}_2(k)$.

Suppose now that $\gamma\in\Gamma$ is a loxodromic element with eigenvalue $\lambda\notin k$. In this case λ satisfies a quadratic polynomial over k; namely, the characteristic polynomial of γ. Thus $k(\lambda)$ is a quadratic field extension of k. Conjugating γ to have the form of its class representative, we may assume that $k(\lambda)\subset A_0\Gamma$. Proposition 3.23 now implies that $A_0\Gamma\subset A_0\Gamma\otimes_k k(\lambda)\cong\mathrm{M}_2(k(\lambda))$, hence Γ is conjugate to a subgroup of $\mathrm{SL}_2(k(\lambda))$. □

The following applications of this corollary are worth noting.

Corollary 4.6. *If Γ is a non-elementary subgroup of $\mathrm{SL}_2(\mathbb{C})$ such that $\mathbb{Q}(\operatorname{tr}\Gamma)\subset\mathbb{R}$, then Γ is conjugate to a subgroup of $\mathrm{SL}_2(\mathbb{R})$.*

Proof. Taking γ to be loxodromic we see that it will have a real trace, hence γ will be hyperbolic. This implies that $\lambda\in\mathbb{R}$ so that the result follows from Corollary 4.5. □

Corollary 4.7. *If Γ is a torsion-free cocompact Kleinian group with finite covolume then Γ contains infinitely many elements which are loxodromic but not hyperbolic.*

Proof. Our hypotheses imply that every element of Γ is loxodromic. Suppose that $\gamma_1,\dots,\gamma_n$ are the only elements of Γ which are not hyperbolic. Because Γ is the fundamental group of a hyperbolic 3-manifold, Γ is *residually finite*; that is, given any non-identity element $\gamma \in \Gamma$ there exists a normal subgroup Γ' of Γ such that $[\Gamma : \Gamma'] < \infty$ and $\gamma \notin \Gamma'$. It follows that there exists a finite index subgroup Γ_0 of Γ which contains none of the elements $\gamma_1, \ldots, \gamma_n$. Every element of Γ_0 is thus hyperbolic so that $\mathbb{Q}(\operatorname{tr}\Gamma_0) \subset \mathbb{R}$. By Corollary 4.6, this means that $\Gamma_0 \subset \mathrm{SL}_2(\mathbb{R})$ and hence has infinite covolume. But this contradicts the fact that $\operatorname{covol}(\Gamma_0) = [\Gamma : \Gamma_0] \cdot \operatorname{covol}(\Gamma) < \infty$. □

Corollary 4.8. *Let Γ be a non-elementary Kleinian group with finite covolume. If an element $\gamma \in \Gamma$ has order greater than $8[k : \mathbb{Q}]^2$ then γ has infinite order.*

Proof. Suppose that $\gamma \in \Gamma$ has finite order m and let λ be the eigenvalue of γ. The field $k(\lambda)$ contains the cyclotomic field $\mathbb{Q}(\zeta_{2m})$ and hence has degree over $\mathbb{Q}$ at least $\varphi(2m)$ where φ is the Euler phi-function. (The $2m$ is because we are looking at a preimage of γ under the projection map $P : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{PSL}_2(\mathbb{C})$.) We have already seen that $2[k : \mathbb{Q}] \geq [k(\lambda) : \mathbb{Q}]$, allowing us to deduce that

$$2[k : \mathbb{Q}] \geq \varphi(2m) \geq \frac{\sqrt{2m}}{2}.$$

The result follows. □

4.2. **Invariant trace field and quaternion algebra.** In the last section we saw that one can attach to any Kleinian group having finite covolume a trace field $\mathbb{Q}(\operatorname{tr}\Gamma)$ and quaternion algebra $A_0\Gamma$ defined over this field. Moreover, we saw that the field $\mathbb{Q}(\operatorname{tr}\Gamma)$ is a finite degree extension of $\mathbb{Q}$, hence we can study the algebra $A_0\Gamma$ using techniques from algebraic number theory. While the trace field and quaternion algebra are invariants of the Kleinian group Γ, they are not in general invariants of the commensurability class of Γ.

Consider for example the subgroup $\Gamma = \langle g, h \rangle$ of $\mathrm{PSL}_2(\mathbb{C})$ where

$$g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix}$$

for $\omega = \frac{-1+\sqrt{-3}}{2}$. All of the matrix entries of Γ lie in the ring of integers $\mathbb{Z}[\omega]$ of $\mathbb{Q}(\sqrt{-3})$, and in fact Γ is a subgroup of the Bianchi group $\mathrm{PSL}_2(\mathbb{Z}[\omega])$ of index 12. Therefore Γ is discrete and $\mathbb{Q}(\operatorname{tr}\Gamma) = \mathbb{Q}(\sqrt{-3})$. Let $x = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ so that the image Px of x in $\mathrm{PSL}_2(\mathbb{C})$ normalizes Γ and has square equal to the identity. The group $\Gamma' = \langle \Gamma, Px \rangle$ is therefore a subgroup of the normalizer of Γ which contains Γ with index 2. In particular Γ and Γ' are directly commensurable. Notice however that Γ' contains the element $xhg = \begin{pmatrix} i & i \\ i\omega & -i + i\omega \end{pmatrix}$. It follows that the trace field of Γ' contains both ω and i and is in fact equal to $\mathbb{Q}(i, \omega)$. Thus Γ and Γ' are commensurable yet have different trace fields.

The above example shows that commensurable groups need not have the same trace field. We will now show how one may associate to any finitely generated non-elementary Kleinian group Γ a subgroup of finite index whose trace field *is* an invariant of its commensurability class.

Definition 4.9. *Let $\Gamma^{(2)} = \langle \gamma^2 : \gamma \in \Gamma \rangle$ be the subgroup of Γ generated by squares.*

Lemma 4.10. *The group $\Gamma^{(2)}$ is a finite index normal subgroup of Γ whose quotient is an elementary abelian 2-group.*

Proof. The group $\Gamma^{(2)}$ is clearly normal in Γ with finite index. It is also clear that every element in the quotient group $\Gamma/\Gamma^{(2)}$ has order 2. The lemma now follows from the fact that Γ, and hence $\Gamma/\Gamma^{(2)}$, is finitely generated. $\square$

Remark 4.11. *Implicit in the proof of Lemma 4.10 was the (very easy to prove) fact that a finite group G in which every element has order 2 is abelian.*

Given a torsion-free Kleinian group Γ with finite covolume one can derive upper bounds on the order of the finite group $\Gamma/\Gamma^{(2)}$ in terms of the covolume of Γ.

Proposition 4.12. *Let Γ be a torsion-free Kleinian group with finite covolume V. There is a constant $c > 0$ such that $[\Gamma : \Gamma^{(2)}] \leq 2^{cV}$.*

Proof. Let $r(\Gamma)$ denote the number of generators of Γ so that $[\Gamma : \Gamma^{(2)}] \leq 2^{r(\Gamma)}$. The proposition now follows from the rank-volume inequality of Gelander [18, Theorem 1.7], which asserts the existence of a constant $c > 0$ for which $r(\Gamma) \leq cV$. $\square$

We are now able to prove this section's main result, that the trace field of $\Gamma^{(2)}$ is an invariant of the commensurability class of Γ. Our proof will make use of the following lemma.

Lemma 4.13. *Let Γ be a finitely generated non-elementary subgroup of $\mathrm{PSL}_2(\mathbb{C})$. If there is a containment $\mathbb{Q}(\operatorname{tr}\Gamma^{(2)}) \subset \mathbb{Q}(\operatorname{tr}\Gamma_1)$ for every finite index subgroup Γ_1 of Γ, then $\mathbb{Q}(\operatorname{tr}\Gamma^{(2)})$ is an invariant of the commensurability class of Γ.*

Proof. Suppose that Δ is commensurable with Γ. Lemma 4.10 shows that $\Gamma^{(2)}$ and $\Delta^{(2)}$ are commensurable, so $\Gamma^{(2)} \cap \Delta^{(2)}$ has finite index in both Γ and Δ. By hypothesis we have inclusions:

- $\mathbb{Q}(\operatorname{tr}\Gamma^{(2)}) \subset \mathbb{Q}(\operatorname{tr}\Gamma^{(2)} \cap \Delta^{(2)})$, and
- $\mathbb{Q}(\operatorname{tr}\Delta^{(2)}) \subset \mathbb{Q}(\operatorname{tr}\Gamma^{(2)} \cap \Delta^{(2)})$.

By definition $\mathbb{Q}(\operatorname{tr}\Gamma^{(2)} \cap \Delta^{(2)}) \subset \mathbb{Q}(\operatorname{tr}\Gamma^{(2)})$ and $\mathbb{Q}(\operatorname{tr}\Gamma^{(2)} \cap \Delta^{(2)}) \subset \mathbb{Q}(\operatorname{tr}\Delta^{(2)})$, so these inclusions are equalities. This shows that $\mathbb{Q}(\operatorname{tr}\Gamma^{(2)}) = \mathbb{Q}(\operatorname{tr}\Delta^{(2)})$. $\square$

Theorem 4.14. *Let Γ be a finitely generated non-elementary subgroup of $\mathrm{PSL}_2(\mathbb{C})$. The field $\mathbb{Q}(\operatorname{tr}\Gamma^{(2)})$ is an invariant of the commensurability class of Γ.*

Proof. By Lemma 4.13 it suffices to consider a finite index subgroup Γ_1 of Γ and show that $\mathbb{Q}(\operatorname{tr}\Gamma^{(2)}) \subset \mathbb{Q}(\operatorname{tr}\Gamma_1)$. In fact, we may even assume that Γ_1 is a normal subgroup of Γ with finite index. This is because if we denote by C the *core* of Γ_1 in Γ (i.e., the intersection of all conjugates of Γ_1 with Γ), then C is a normal subgroup of Γ with finite index and satisfies $\mathbb{Q}(\operatorname{tr} C) \subset \mathbb{Q}(\operatorname{tr}\Gamma_1)$. It therefore suffices to show that $\mathbb{Q}(\operatorname{tr}\Gamma^{(2)}) \subset \mathbb{Q}(\operatorname{tr} C)$.

The argument in the previous paragraph shows that we may assume Γ_1 is a finite index normal subgroup of Γ. We now claim that if $g \in \Gamma$ then $g^2 \in A_0\Gamma_1$. Because we are assuming that Γ_1 is normal in Γ, conjugation by g induces an automorphism of Γ_1 and hence of $A_0\Gamma_1$. The Skolem-Noether theorem (Theorem 3.15) implies that every automorphism of a quaternion algebra is inner, hence there exists an element $a \in (A_0\Gamma_1)^*$ such that the induced automorphism is given by

$$x \mapsto axa^{-1}$$

for all $x \in A_0\Gamma_1$. Considering $A_0\Gamma_1 \otimes \mathbb{C} = \mathrm{M}_2(\mathbb{C})$, we claim that $g^{-1}a$ is central and hence $g^{-1}a = y\,\mathrm{Id}$ for some $y \in \mathbb{C}$. Indeed, it suffices to show that $g^{-1}a$ commutes with every element $\gamma \in \Gamma$. But this is clear because in this context we have, by definition of a, that $a\gamma a^{-1} = g\gamma g^{-1}$. Consequently $g^{-1}a = y\,\mathrm{Id}$ and

$$y^2 = \det(y\,\mathrm{Id}) = \det(g^{-1}a) = \det(g^{-1})\det(a) = \det(a)$$

so that $y^2 = \det(a) \in \mathbb{Q}(\operatorname{tr}\Gamma_1)$ and $g^2 = y^{-2}a^2 \in A_0\Gamma_1$ as claimed. Since g was an arbitrary element of Γ we conclude that $\Gamma^{(2)} \subset A_0\Gamma_1$, hence $\operatorname{tr}\Gamma^{(2)} \subset \operatorname{tr} A_0\Gamma_1 \subset \mathbb{Q}(\operatorname{tr}\Gamma_1)$ and consequently $\mathbb{Q}(\operatorname{tr}\Gamma^{(2)}) \subset \mathbb{Q}(\operatorname{tr}\Gamma_1)$. □

Corollary 4.15. *Let Γ be a finitely generated non-elementary subgroup of $\mathrm{PSL}_2(\mathbb{C})$. The quaternion algebra $A_0\Gamma^{(2)}$ is an invariant of the commensurability class of Γ.*

Proof. Suppose that Γ and Δ are commensurable so that $\mathbb{Q}(\operatorname{tr}\Gamma^{(2)}) = \mathbb{Q}(\operatorname{tr}\Delta^{(2)})$. Let $x, y \in \Gamma^{(2)} \cap \Delta^{(2)}$ be loxodromic elements for which $\langle x, y\rangle$ is irreducible. Then

$$A_0\Gamma^{(2)} = \mathbb{Q}(\operatorname{tr}\Gamma^{(2)})[\mathrm{Id}, x, y, xy] = \mathbb{Q}(\operatorname{tr}\Delta^{(2)})[\mathrm{Id}, x, y, xy] = A_0\Delta^{(2)}$$

by Corollary 4.4. □

Definition 4.16. *Let Γ be a finitely generated non-elementary subgroup of $\mathrm{PSL}_2(\mathbb{C})$. The field $\mathbb{Q}(\operatorname{tr}\Gamma^{(2)})$ will henceforth be denoted by $k\Gamma$ and referred to as the* **invariant trace field** *of Γ. Similarly, the quaternion algebra $A_0\Gamma^{(2)}$ will be denoted $A\Gamma$ and referred to as the* **invariant quaternion algebra** *of Γ.*

Caution 4.17. *Although the invariant trace field and quaternion algebra are indeed commensurability invariants, they are not* **complete commensurability invariants***; that is, there are examples of non-commensurable manifolds with the same invariant trace field and quaternion algebra. We will see however, that these are complete commensurability invariants when the manifold is arithmetic.*

Maclachlan and Reid [24, Theorem 3.6.2] were able to employ Corollary 4.4 so as to provide the following explicit description of $A\Gamma$.

Theorem 4.18. *If g and h are elements of the non-elementary Kleinian group Γ such that $\langle g, h\rangle$ is irreducible, g, h do not have order 2 in $\mathrm{PSL}_2(\mathbb{C})$ and g is not parabolic, then*

$$A\Gamma = \left(\frac{\operatorname{tr}^2 g - 4, \operatorname{tr}[g,h] - 2}{k\Gamma}\right).$$

We now prove a result which constrains the number fields which may occur as the invariant trace field of a finite covolume Kleinian group. Later on, under the assumption of arithmeticity, we will give a precise classification of the number fields arising as invariant trace fields.

Corollary 4.19. *If Γ is a Kleinian group with finite covolume then its invariant trace field $k\Gamma$ is a non-real extension of $\mathbb{Q}$ of finite degree.*

Proof. That $k\Gamma$ is a finite extension of $\mathbb{Q}$ is an immediate consequence of Theorem 4.2. Were $k\Gamma$ to be contained in $\mathbb{R}$, $\Gamma^{(2)}$ would be conjugate to a subgroup of $\mathrm{SL}_2(\mathbb{R})$, contradicting the fact that $\Gamma^{(2)}$ has finite covolume. □

We are now in a position to make our first connection between the topology of hyperbolic manifolds and the structure of their quaternion algebras.

Theorem 4.20. *If* Γ *is a Kleinian group for which* $\mathbf{H}^3/\Gamma$ *is non-compact, then* $A\Gamma = \mathrm{M}_2(k\Gamma)$.

Proof. If Γ contains a parabolic element then so does $\Gamma^{(2)}$. Let $\gamma \in \Gamma^{(2)}$ be parabolic. Then $\gamma - \mathrm{Id}$ is not invertible in $A\Gamma$. This implies that $A\Gamma$ cannot be a division algebra so that the result follows from Wedderburn's theorem (Theorem 3.17). □

4.3. **Application I: the Weeks manifold.** The work of Jørgensen and Thurston shows that the set of volumes of compact hyperbolic 3-manifolds is well-ordered. In particular there exists a compact hyperbolic 3-manifold of smallest volume. Until fairly recently the identity of this smallest volume compact hyperbolic 3-manifold was unknown. In [27] Meyerhoff used Jørgensen's theorem (Theorem 2.10) to derive a lower bound for the volume of a compact hyperbolic 3-manifold. Such a manifold must have volume greater than 0.00064. In the same paper he suggested that the manifold manifold arising from $(5, 1)$ Dehn surgery on the figure eight knot in the 3-sphere could very well be the sought after manifold. This manifold, now known as the *Meyerhoff manifold*, is known to have volume 0.9812... The Meyerhoff manifold was proven to be arithmetic by Chinburg [7].

It is now known however that the compact hyperbolic 3-manifold of smallest volume is the *Weeks manifold*. The compact hyperbolic 3-manifold of second smallest volume is the Meyerhoff manifold. The Weeks manifold has volume 0.9427... and arises from $(5, 2)$ and $(5, 1)$ surgery on the Whitehead link. (The Weeks manifold is named after Jeffrey Weeks who discovered it in his 1985 Ph.D. thesis [37].) That the Weeks manifold has smallest volume amongst all compact hyperbolic 3-manifolds was proven in 2009 by Gabai, Meyerhoff and Milley [17]. It had been proven earlier by Chinburg, Friedman, Jones and Reid [9] that the Weeks manifold is arithmetic and is in fact the arithmetic hyperbolic 3-manifold of smallest volume.

Remark 4.21. *Observe that the two smallest volumes achieved by compact hyperbolic* 3*-manifolds are both achieved by arithmetic manifolds. This is true in dimension* 2 *as well and is considered likely to be true in general. Of all known examples of compact hyperbolic* 4*-manifolds for instance, the one with the smallest volume is the* Davis manifold [11, 16, 30]*. The Davis manifold has Euler characteristic* $\chi(M) = 26$*, hence volume* $V = 4\pi^2\chi(M)/3$*, and is known to be arithmetic.*

In this section we will describe the Weeks manifold in more detail and compute its invariant trace field and quaternion algebra. Our proof follows Chinburg [7] and Chinburg, Friedman, Jones and Reid [9, pp. 24-25].

Let M_W denote the Weeks manifold and set $\pi_1(M_W)$ so that $M_W = \mathbf{H}^3/\Gamma$. It is known that $\pi_1(M)$ has presentation

$$\pi_1(M_W) = \langle a, b : a^2b^2a^2b^{-1}ab^{-1} = a^2b^2a^{-1}ba^{-1}b^2 = 1\rangle. \tag{2}$$

Lemma 4.22. *There is a representation* $\rho : \pi_1(M_W) \to \mathrm{SL}_2(\mathbb{C})$ *such that the induced projective representation* $\overline{\rho} : \pi_1(M_W) \to \mathrm{PSL}_2(\mathbb{C})$ *is discrete, faithful, and satisfies* $M_W = \mathbf{H}^3/\overline{\rho}(\pi_1(M_W))$*. Let* $A = \rho(a)$ *and* $B = \rho(b)$*. There are nonzero* $x, y, r \in \mathbb{C}$ *with* $|x|, |y| \neq 1$ *and*

$$A = \begin{pmatrix} x & 1 \\ 0 & x^{-1} \end{pmatrix}, \qquad B = \begin{pmatrix} y & 0 \\ r & y^{-1} \end{pmatrix}.$$

Proof. Because M_W is an orientable hyperbolic 3-manifold its hyperbolic structure arises from some faithful discrete representation $\rho_1 : \pi_1(M_W) \to \mathrm{PSL}_2(\mathbb{C})$. Let A, B be elements of $\mathrm{SL}_2(\mathbb{C})$ whose images in $\mathrm{PSL}_2(\mathbb{C})$ satisfy $PA = \rho_1(a)$ and $PB = \rho_1(b)$. Define $f(a,b) = a^2b^2a^2b^{-1}ab^{-1}$ and $g(a,b) = a^2b^2a^{-1}ba^{-1}b^2$ so that

$$\pi_1(M_W) = \langle a, b : f(a,b) = g(a,b) = 1\rangle.$$

Then $f(A,B) = \pm\,\mathrm{Id}$ and $g(A,B) = \pm\,\mathrm{Id}$. Multiplying A and B by $\pm\,\mathrm{Id}$ as needed, we can take $f(A,B) = 1 = g(A,B)$. This gives us a lifting $\rho_2 : \pi_1(M_W) \to \mathrm{SL}_2(\mathbb{C})$.

We claim that A and B have no common nonzero eigenvectors. Suppose to the contrary that A and B do contain a common nonzero eigenvector. This implies that $(AB)^{-1}BA$ is unipotent. But $\pi_1(M_W)$ contains no parabolic elements since M_W is compact, implying that $(ab)^{-1}ba = 1$. But this implies that a and b commute, which in turn implies that $\pi_1(M_W)$ is abelian. This contradiction proves our claim.

There exists a basis $\{v_1, v_2\}$ of $\mathbb{C}^2$ such that $Av_1 = xv_1$ and $Bv_2 = y^{-1}v_2$ for nonzero $x, y \in \mathbb{C}$ with $|x|, |y| \neq 1$. The previous paragraph shows that v_2 is not an eigenvector of A, hence we may multiply v_1 by some non-zero scalar so as to have $Av_2 = x^{-1}v_2 + v_1$. Relative to the basis $\{v_1, v_2\}$, A and B now have the required form for some $r \in \mathbb{C}$. Notice that $r \neq 0$, for otherwise v_1 would be a common eigenvector of both A and B.

The resulting representation $\rho : \pi_1(M_W) \to \mathrm{SL}_2(\mathbb{C})$ is conjugate to ρ_2 and has the requisite properties, proving the lemma. □

We now write the first relation for $\pi_1(M_W)$ as $w = 0$ where $w = w(a,b) = a^2b^2a^2 - ba^{-1}b$. The same relation must hold for A, B as in Lemma 4.22, giving us a system of 4 equations in x, y, r: $w_{11} = w_{12} = w_{21} = w_{22} = 0$, where w_{ij} corresponds to the (i,j)th matrix entry in $w(A,B)$. For instance we have

$$0 = w_{21} = r(x - x^2 + rxy - y^2 + xy^2).$$

Because $r \neq 0$ we must have

$$r = \frac{x^2 - x + y^2 - xy^2}{xy}. \tag{3}$$

Upon substituting (3) into w_{11}, w_{12}, w_{22}, we see that w_{11}, w_{12}, w_{22} are all divisible by

$$p(x,y) = 1 + x^2 + y^2 - xy^2 + x^2y^2 + y^4 + x^2y^4$$

and that $w_{11} = w_{12} = w_{22} = 0$ only if $p(x,y) = 0$.

We now write the second relation as $u = 0$ where $u = a^2b^2 - b^{-2}ab^{-1}a$. Solving for $u = 0$ as above we deduce that

$$u_{12} = (x - y)(xy - 1) = 0,$$

whence $x = y$ or $x = y^{-1}$. Suppose that $x = y$ (the other case will yield the same conclusion) so that $p(x,y) = p(x,x) = 1 + 2x^2 - x^3 + 2x^4 + x^6$. Solving for $z = x + x^{-1}$ yields $z^3 - z - 1 = 0$, and by equation (3) we must have $r = 2 - z$.

Let $\Gamma = \pi_1(M_W)$. We claim that the invariant trace field $k\Gamma$ of M_W is $\mathbb{Q}(z)$ where $z^3 - z - 1 = 0$. Indeed, it is clear that the trace field of Γ is $\mathbb{Q}(z)$, hence $k\Gamma \subset \mathbb{Q}(z)$. As $\mathbb{Q}(z)$ has degree 3 over $\mathbb{Q}$ and this no proper subfields aside from $\mathbb{Q}$, it suffices to show that $k\Gamma \neq \mathbb{Q}$. Consider the element $\mathrm{tr}\, A^2 = (x^2 + x^{-2}) \in k\Gamma$. We will show that $\mathbb{Q}(\mathrm{tr}\, A^2) \neq \mathbb{Q}$. Suppose therefore to the contrary that $x^2 + x^{-2} \in \mathbb{Q}$ and consider

$z^2 = (x + x^{-1})^2 = 2 + x^2 + x^{-2}$. In this case $\mathbb{Q}(z)$ could have degree at most 2 which is a contradiction. This proves that $k\Gamma = \mathbb{Q}(z)$. We note that an alternative proof that $k\Gamma = \mathbb{Q}(z)$ could have been obtained by using the presentation (2) of $\pi_1(M_W)$ to show that $\Gamma^{(2)} = \Gamma$ (i.e., that $a, b \in \Gamma^{(2)}$). Similar considerations show that the set $\{\operatorname{tr}\gamma : \gamma \in \Gamma\}$ consists entirely of algebraic integers in $\mathbb{Q}(z)$.

The invariant quaternion algebra $A\Gamma$ of M_W is given by the Hilbert symbol in Theorem 4.18. Using the Magma [5] computer algebra system, we calculate that $A\Gamma$ is the quaternion algebra over $k\Gamma = \mathbb{Q}(z)$ which is ramified at the prime of norm 5 and the unique real place of $\mathbb{Q}(z)$. The relevant commands are:

```
> R<z>:=PolynomialRing(Rationals());
> F<a>=NumberField(z^3-z-1);
> A:=QuaternionAlgebra<F | z^2-4,3*z^2-z-5>;
> A;
Quaternion Algebra with base ring F, defined by i^2 = -3*z^2 + z, j^2 =
-3*z2 + 2
> RamifiedPlaces(A);
[ Prime Ideal
Two element generators:
[5, 0, 0]
[8, 1, 0]]
[ 1st place at infinity ]
> Norm(RamifiedPlaces(A)[1]);
5
```

The above discussion allows us to conclude the following.

Theorem 4.23. *Let M_W be the Weeks manifold and $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ its fundamental group.*

(1) The invariant trace field $k\Gamma$ of M_W is $\mathbb{Q}(z)$ where $z^3 - z - 1 = 0$. This field has discriminant -23 and precisely one complex place.
(2) $\operatorname{tr}\gamma$ is an algebraic integer for all $\gamma \in \Gamma$.
(3) $A\Gamma$ is the quaternion algebra over $k\Gamma$ which is ramified at the unique real place of $k\Gamma$ and the prime ideal lying above the rational prime 5.

Chinburg's proof of the arithmeticity of the Meyerhoff manifold was carried out by determining its invariant trace field and quaternion algebra and proving a theorem analogous to Theorem 4.23. We leave this as an exercise for the interested reader.

Exercise 4.24. *It is known that the fundamental group of the Meyerhoff manifold has presentation*

$$\langle a, b : (ab^{-1}a^{-1}b)a(ab^{-1}a^{-1}b)^{-1}b^{-1} = a^4b^{-1}aba^{-2}bab^{-1}a = 1\rangle.$$

Compute the invariant trace field and quaternion algebra of the Meyerhoff manifold.

4.4. **Application II: manifolds with no totally geodesic surfaces.** In this section we will prove a theorem which relates the geometry of a finite-volume hyperbolic 3-manifold to the arithmetic structure of its invariant trace field and quaternion algebra.

Theorem 4.25. *Let Γ be a Kleinian group of finite covolume which satisfies the following two conditions.*

(1) $k\Gamma$ contains no proper subfield other than $\mathbb{Q}$.

(2) $A\Gamma$ ramifies at at least one infinite place of $k\Gamma$.

Then Γ contains no hyperbolic elements.

Proof. As Γ contains a hyperbolic element if and only if $\Gamma^{(2)}$ does, it suffices to consider $\Gamma^{(2)}$. Let $\gamma \in \Gamma^{(2)}$ be a hyperbolic element. As $\operatorname{tr}\gamma \in \mathbb{R}$ we see that $\operatorname{tr}\gamma \in k\Gamma \cap \mathbb{R}_{>2} = \mathbb{Q}_{>2}$.

Suppose now that $A\Gamma$ ramifies at some infinite place ν of $k\Gamma$, which of course is necessarily real. Thus $A\Gamma \otimes_{k\Gamma} k\Gamma_\nu \cong \mathbb{H}$. Let $\sigma : k\Gamma \hookrightarrow \mathbb{R}$ be the embedding associated to ν and let $\psi : A\Gamma \hookrightarrow \mathbb{H}$ be an extension of σ (i.e., $\psi(x) = \sigma(x)$ for all $x \in k\Gamma$). Then

$$\psi(\Gamma^{(2)}) \subset \psi(A\Gamma^1) \subset \mathbb{H}^1,$$

where superscript 1 indicates the multiplicative subgroup of elements with reduced norm 1. As $\operatorname{tr}\gamma \in \mathbb{Q}$,

$$\operatorname{tr}\gamma = \sigma(\operatorname{tr}\gamma) = \psi(\gamma + \overline{\gamma}) = \psi(\gamma) + \overline{\psi(\gamma)} = \operatorname{tr}\psi(\gamma).$$

We conclude that $\operatorname{tr}\gamma \in \operatorname{tr}\mathbb{H}^1$. But this is a contradiction because $\operatorname{tr}\mathbb{H}^1 = [-2, 2]$ and $|\operatorname{tr}\gamma| > 2$. ☐

We now record two important corollaries of Theorem 4.25, the first geometric and the second group theoretic.

Corollary 4.26. *If $M = \mathbf{H}^3/\Gamma$ is a finite volume hyperbolic 3-manifold for which Γ satisfies the conditions of Theorem 4.25 then M contains no immersed totally geodesic surfaces.*

Corollary 4.27. *If Γ is a Kleinian group of finite covolume which satisfies the conditions of Theorem 4.25 then Γ contains no non-elementary Fuchsian groups.*

Hyperbolic 3-manifolds whose fundamental groups satisfy the conditions of Theorem 4.25 are actually quite common. Consider the Weeks manifold, defined in Section 4.3. In Theorem 4.23 we saw that the invariant trace field of the Weeks manifold is a cubic number field with precisely one real place and one complex place. Condition (1) of Theorem 4.25 is therefore satisfied. We also saw that the invariant quaternion algebra of the Weeks manifold is ramified at the unique real place of $k\Gamma$, which means that condition (2) is satisfied as well. We therefore conclude the following.

Corollary 4.28. *The Weeks manifold contains no immersed totally geodesic surfaces.*

5. Orders in quaternion algebras: a first glimpse

In this section we introduce orders in quaternion algebras and explore some of their basic properties. Our goal is to provide the background necessary to describe the construction of discrete subgroups of $\mathrm{PSL}_2(\mathbb{C})$ from orders in quaternion algebras defined over certain number fields. This will allow us to give (finally!) the definition of an arithmetic hyperbolic 3-manifold.

5.1. Defining orders.

Let R be a Dedekind domain (i.e., R is an integral domain which is noetherian, integrally closed and in which every prime ideal is maximal) with quotient field K. In practice we will always take K to be a number field or its completion at a finite prime and R to be its ring of integers. Let A be a quaternion algebra over K.

Definition 5.1. *An element $\alpha \in A$ is* **integral** *with respect to R if its (reduced) characteristic polynomial $x^2 - \mathrm{tr}(\alpha)x + n(\alpha)$ has coefficients in R. We call $\mathrm{tr}(\alpha)$ the (reduced)* **trace** *of α and $n(\alpha)$ the (reduced)* **norm** *of α.*

The reduced trace and norm of an element $\alpha \in A$ may be defined in a somewhat different manner, as follows.

Let $\{1, i, j, ij\}$ be the standard basis of A. Given an element $\alpha \in A$ we may write

$$\alpha = a_0 + a_1 i + a_2 j + a_3 ij$$

for $a_0, a_1, a_2, a_3 \in K$.

Definition 5.2. *Let A_0 be the subspace of A spanned by $\{i, j, ij\}$. The elements of A_0 are called* **pure quaternions***.*

It is easy to check that the pure quaternions of A are characterized by the property that they do not lie in K but their squares do. (In other words an element $x \in A$ lies in A_0 if and only if $x \notin K$ but $x^2 \in K$.) In particular the property of being a pure quaternion does not depend on the choice of basis. We may therefore uniquely decompose any element $\alpha \in A$ as $\alpha = a_0 + \alpha'$ where $a_0 \in K$ and $\alpha' \in A_0$.

Definition 5.3. *Given an element $\alpha \in A$, we define the* **conjugate** *of α to be $\overline{\alpha} = a_0 - \alpha'$.*

The conjugation operation just defined equips A with an anti-involution such that

(1) $\overline{x+y} = \overline{x} + \overline{y}$;
(2) $\overline{xy} = \overline{y}\,\overline{x}$;
(3) $\overline{\overline{x}} = x$;
(4) $\overline{rx} = r\overline{x}$ for all $r \in R$;
(5) If $A = \mathrm{M}_2(K)$ then

$$\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Definition 5.4. *For $\alpha \in A$ we define the* **reduced trace** *$\mathrm{tr}(\alpha)$ to be $\mathrm{tr}(\alpha) = \alpha + \overline{\alpha}$ and the* **reduced norm** *$n(\alpha)$ to be $n(\alpha) = \alpha\overline{\alpha}$.*

Suppose now that $\{1, i, j, ij\}$ is a standard basis for A where $i^2 = a$ and $j^2 = b$. Then if $\alpha = a_0 + a_1 i + a_2 j + a_3 ij$ we see that

$$\mathrm{tr}(\alpha) = 2a_0$$

and

$$n(\alpha) = a_0^2 - aa_1^2 - ba_2^2 + aba_3^2.$$

This means that α is integral precisely when $2a_0, a_0^2 - aa_1^2 - ba_2^2 + aba_3^2 \in R$.

Recall that the set of all integral elements of a number field has the structure of a ring (and importantly for what follows, a finitely generated $\mathbb{Z}$-module). This is certainly not the case for elements of quaternion algebras. Consider the following two elements of $\mathrm{M}_2(\mathbb{Q})$:

$$A = \begin{pmatrix} \frac{5}{4} & -\frac{1}{8} \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix}, \qquad B = \begin{pmatrix} \frac{11}{6} & \frac{1}{2} \\ \frac{5}{18} & \frac{7}{6} \end{pmatrix}.$$

The characteristic polynomials of A and B are $p_A(x) = x^2 - 2x + 1$ and $p_B(x) = x^2 - 3x + 2$. Thus both A and B are integral (with respect to $\mathbb{Z}$). Neither $A + B$ nor AB are integral however; their characteristic polynomials are $p_{A+B}(x) = x^2 - 5x + \frac{809}{144}$ and $p_{AB}(x) = x^2 - \frac{487}{144}x + 2$. We shall see that the failure of the set of integral elements in a quaternion algebra to be a ring makes the theory of orders in quaternion algebras significantly more complicated than the study of orders in number fields. On the other hand, this failure also makes the theory much richer. Indeed, it is precisely what makes Vignéras construction of isospectral Riemann surfaces and hyperbolic 3-manifolds possible.

Definition 5.5. *Let V be a vector space over K. An R-**lattice** in V is a finitely generated R-module contained in V. An R-lattice L is **complete** if $L \otimes_R K \cong V$.*

The following is a basic result in commutative algebra.

Proposition 5.6 (Atiyah-Macdonald [2, Prop. 5.1])**.** *An element $\alpha \in A$ is integral if and only if $R[\alpha]$ is an R-lattice in A.*

We are now able to provide our first definition of orders in quaternion algebras.

Definition 5.7. *An **order** $\mathcal{O}$ in A is a complete R-lattice in A which is also a subring of A. A **maximal order** is an order in A which is maximal with respect to inclusion.*

Example 5.8. We give a few examples of orders.

(1) The ring $\mathrm{M}_2(R)$ is always an order of $\mathrm{M}_2(K)$.
(2) Suppose that $A = \left(\frac{a,b}{K}\right)$ where a, b are integral elements of K. (Note that A can always be put in this form as the Hilbert symbol is only defined up to squares, meaning that we can 'clear denominators' by multiplying a, b by a square element of K.) Then $R[1, i, j, ij]$ is an order of A.
(3) Given a complete R-lattice I in A, the *left and right orders* of I are

$$\mathcal{O}_\ell(I) = \{\alpha \in A : \alpha I \subset I\},$$
$$\mathcal{O}_r(I) = \{\alpha \in A : I\alpha \subset I\}.$$

Proposition 5.9. *$\mathcal{O}$ is an order in A if and only if $\mathcal{O}$ is a ring of integral elements in A which contains R and satisfies $\mathcal{O} \otimes_R K = A$. Moreover, every order is contained in a maximal order.*

Proof. Let $\mathcal{O}$ be an order of A and $\alpha \in \mathcal{O}$. Since $\mathcal{O}$ is an R-lattice, so is $R[\alpha]$. It now follows from Proposition 5.6 that α is integral. That $\mathcal{O}$ satisfies the other properties follows from our definition of order.

We now prove the converse. Because $\mathcal{O} \otimes_R K = A$ we may choose a basis $\{x_1, x_2, x_3, x_4\}$ of A for which all of the x_i lie in $\mathcal{O}$. As the reduced trace determines a non-singular symmetric bilinear form on A, $d = \det(\operatorname{tr}(x_i x_j)) \neq 0$. Set $L = \{\sum a_i x_i : a_i \in R\}$. Then $L \subset \mathcal{O}$ because $R \subset \mathcal{O}$ and each of the $x_i \in \mathcal{O}$. Suppose that $\alpha \in \mathcal{O}$ with $\alpha = \sum b_i x_i$ with $b_i \in K$. For each j we have that $\alpha x_j \in \mathcal{O}$, so $\operatorname{tr}(\alpha x_j) = \sum b_i \operatorname{tr}(x_i x_j) \in R$. Therefore $b_i \in \frac{1}{d}R$ and $\mathcal{O} \subset \frac{1}{d}L$. It follows that $\mathcal{O}$ is finitely generated, proving the first assertion. The second assertion follows from a Zorn's lemma argument. □

Lemma 5.10. *The order* $\mathrm{M}_2(R)$ *is a maximal order of* $\mathrm{M}_2(K)$.

Proof. If $\mathrm{M}_2(R)$ is not maximal then let $\mathcal{O}$ be a maximal order containing $\mathrm{M}_2(R)$ and some element $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ with at least one of x, y, z, w not in R. By adding and multiplying elements of R we can produce an element $\alpha \in \mathcal{O}$ where $\alpha = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ and $a \notin R$. Such an element is clearly not integral, a contradiction. □

We have shown that if $\mathcal{O}$ is an order in A then every element of $\mathcal{O}$ is integral. The following proposition provides a converse to this statement.

Proposition 5.11. *If* $\alpha \in A$ *is integral then* α *is contained in a maximal order of* A.

Proof. If $\alpha \in R$ then α is in *every* order of A by Proposition 5.9. We may therefore assume that $\alpha \notin R$, in which case $K(\alpha)$ is a quadratic field extension of K which is contained in A. Let $\beta \in A^*$ be such that $\beta\alpha\beta^{-1} = \overline{\alpha}$. The existence of such an element is due to the Skolem-Noether theorem and we may take β to be integral by simply clearing denominators. The R-module generated by α and β is $R + R\alpha + R\beta + R\alpha\beta$ and is clearly an order of A. This order might not be maximal, but we have seen that every order is contained in a maximal order. □

5.2. **Orders in matrix algebras.** We now consider the special case in which $A = \mathrm{M}_2(K)$. This case turns out to be particularly important in studying orders in quaternion algebras over number fields, as one would like to be able to work locally and at all but finitely many primes a quaternion algebra (over a number field) is split.

Let V be a two-dimensional vector space over K with basis $\{e_1, e_2\}$ so that we may identify A with $\operatorname{End}(V)$. Given a complete R-lattice L in V we define

$$\operatorname{End}(L) = \{\sigma \in \operatorname{End}(V) : \sigma(L) \subset L\}.$$

Consider the complete R-lattice $L_0 = Re_1 + Re_2$, for which we have $\operatorname{End}(L_0) = \mathrm{M}_2(R)$. The lattice L_0 also has the property that if L is a complete R-lattice then there is an element $a \in R$ such that

$$aL_0 \subset L \subset a^{-1}L_0.$$

It follows that

$$a^2 \operatorname{End}(L_0) \subset \operatorname{End}(L) \subset a^{-2} \operatorname{End}(L_0).$$

Therefore each $\operatorname{End}(L)$ is an order. In fact, these are the maximal orders of $\operatorname{End}(V)$.

Lemma 5.12. *If $\mathcal{O}$ is an order in* $\mathrm{End}(V)$ *then there exists a complete R-lattice L of V such that $\mathcal{O} \subset \mathrm{End}(L)$.*

Proof. Let $L_0 = Re_1 + Re_2$ be as above and $L = \{\ell \in L_0 : \mathcal{O}\ell \subset L_0\}$. Then L is an R-submodule of L_0 and in particular is finitely generated. Also, if $a \in R$ is non-zero with $a\,\mathrm{End}(L_0) \subset \mathcal{O} \subset a^{-1}\,\mathrm{End}(L_0)$ then for all $\ell \in L_0$ we have

$$\mathcal{O}a\ell \subset \mathrm{End}(L_0)\ell \subset L_0,$$

hence $a\ell \in L_0$. Thus $aL_0 \subset L$. Therefore L is a complete R-lattice in V.

We now show that $\mathcal{O} \subset \mathrm{End}(L)$. Let $\alpha \in \mathcal{O}$. For all $\ell \in L$ we have $\mathcal{O}a\ell \subset \mathcal{O}\ell \subset L_0$. Therefore $\alpha\ell \in L$ and $\mathcal{O} \subset \mathrm{End}(L)$. □

Corollary 5.13. *If R is a PID then every maximal order in* $\mathrm{M}_2(K)$ *is conjugate to* $\mathrm{M}_2(R)$.

Proof. As above we identify $\mathrm{M}_2(K)$ with $\mathrm{End}(V)$ where V is a two-dimensional vector space over K with basis $\{e_1, e_2\}$. Let $L_0 = Re_1 + Re_2$ and $\mathcal{O}$ be a maximal order of $\mathrm{M}_2(K)$. Lemma 5.12 and the accompanying discussion shows that there exists a complete R-lattice L in V such that $\mathcal{O} = \mathrm{End}(L)$. Let $\{f_1, f_2\}$ be a basis of V such that $L = Rf_1 + Rf_2$ and define an element $\sigma \in \mathrm{End}(V)$ by $\sigma(e_1) = f_1$ and $\sigma(e_2) = f_2$. Then $\sigma(L_0) = L$ and $\mathrm{End}(L) = \sigma\,\mathrm{End}(L_0)\sigma^{-1}$; that is, $\mathrm{End}(L) = \sigma\,\mathrm{M}_2(R)\sigma^{-1}$. In particular $\mathcal{O}$ is conjugate to $\mathrm{M}_2(R)$. □

Corollary 5.13 will be vitally important when we study orders in matrix algebras over non-archimedean local fields, as the ring of integers of such a field is always a PID. The corollary is also useful for studying quaternion orders over number fields however. It implies for example, that $\mathrm{M}_2(\mathbb{Z})$ is a maximal order of $\mathrm{M}_2(\mathbb{Q})$.

When R is not a PID it is no longer the case that every complete R-lattice L is free as an R-module. Something can still be said however, even in this level of generality. First we recall a definition.

Definition 5.14. *A* **fractional ideal** *I of R is an R-submodule of K such that there exists an element $r \in R$ with $rI \subset R$.*

Suppose now that R is a Dedekind domain which is not necessarily a PID. The structure theorem for finitely generated modules over a Dedekind domain gives us a basis $\{x, y\}$ of V and a fractional ideal I of K such that $L = Rx + Iy$. In terms of matrices this means that $\mathrm{End}(L)$ is conjugate to

$$\mathrm{M}_2(R, I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, d \in R, b \in I^{-1}, c \in I \right\}.$$

5.3. **Orders in local division algebras.** In this section we let $K = k_v$ be a p-adic field, $R = \mathcal{O}_K$ and π be a local uniformizer. We saw in Section 3.5 that over K there are precisely 2 isomorphism classes of quaternion algebras over K: $\mathrm{M}_2(K)$ and $\left(\frac{u,\pi}{K}\right)$, where $K(\sqrt{u})$ is the unique unramified quadratic extension of K. Because R is a PID, Corollary 5.13 implies that every maximal order of $\mathrm{M}_2(K)$ is conjugate to $\mathrm{M}_2(R)$. In this section we will discuss maximal orders in $\left(\frac{u,\pi}{k}\right)$ and find that in this case there is a unique maximal order.

Definition 5.15. *If F is a field then a* **discrete valuation** $\nu : F \longrightarrow \mathbb{Z} \cup \{\infty\}$ *is a surjective map such that for all $x, y \in F$:*

(1) $\nu(x) = \infty$ *if and only if* $x = 0$*;*
(2) $\nu(xy) = \nu(x) + \nu(y)$*;*
(3) $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$.

In the setting we are considering, K is a p-adic field equipped with the exponential p-adic valuation ν_K. Given an element $x \in K$ there is a unique integer m such that $x = u\pi^m$ for some unit $u \in \mathcal{O}_K^*$. In this case $\nu_K(x) = m$.

Our goal is to show that the valuation ν_K on K may be extended to a "non-commutative" valuation on the unique quaternion division algebra $\left(\frac{u,\pi}{K}\right)$ over K. Before doing it is useful to recall that if L/K is a finite extension then L is also a p-adic field and that there is a unique extension of ν_K to a valuation ν_L on L with respect to which L is complete via the formula

$$\nu_L(x) = \frac{1}{f} \cdot \nu_K(N_{L/K}(x)),$$

where $N_{L/K}$ denotes the field norm on the extension L/K and $f = f(L/K)$ the corresponding inertia degree. (Recall that the **inertia degree** is defined to be the degree of the corresponding extension of residue fields: $[\mathcal{O}_L/\pi_L\mathcal{O}_L : \mathcal{O}_K/\pi_K\mathcal{O}_K]$.)

We now define a valuation w on $\left(\frac{u,\pi}{K}\right)$. Let $x \in \left(\frac{u,\pi}{K}\right)$ and define

$$w(x) = \nu_K(n(x)),$$

where $n(x)$ is the reduced norm of x. The restriction of w to $\left(\frac{u,\pi}{K}\right)^*$ gives a homomorphism to $\mathbb{Z}$ such that $w|_K = 2\nu_K$, as

$$w(y) = w(n(y)) = \nu_K(y^2) = \nu_K(y) + \nu_K(y) = 2\nu_K(y)$$

for all $y \in K$. Here we have used the fact that the reduced norm on $\left(\frac{u,\pi}{K}\right)$ is multiplicative. Moreover, if L/K is a quadratic field extension and $L \subset \left(\frac{u,\pi}{K}\right)$ then the reduced norm of an element $\ell \in L$ coincides with the field norm $N_{L/K}(\ell)$. The restriction of w to L therefore defines a valuation on L which is equivalent to the extended valuation ν_L. In particular if we denote by $\mathcal{O}$ the set of elements of $\left(\frac{u,\pi}{k}\right)$ whose valuations with respect to w are non-negative, then $x \in \mathcal{O}$ implies that $x \in \mathcal{O}_L$ where $L = K(x)$. Conversely, if $x \in \left(\frac{u,\pi}{K}\right)$ is integral, then $n(x) \in \mathcal{O}_K$, hence $w(x) = \nu_K(n(x)) \geq 0$ and $x \in \mathcal{O}$. It follows that $\mathcal{O}$ is the ring consisting of all integral elements of $\left(\frac{u,\pi}{K}\right)$.

Now suppose that $y \in \left(\frac{u,\pi}{K}\right)$. Writing y in terms of the standard $\{1, i, j, ij\}$ basis and clearing denominators shows that there exists $r \in \mathcal{O}_K$ such that $ry \in \mathcal{O}$, hence

$$\mathcal{O} \otimes_{\mathcal{O}_K} K \cong \left(\frac{u,\pi}{K}\right)$$

and $\mathcal{O}$ is an order by Proposition 5.9. Every order is contained in a maximal order, which by definition may contain only integral elements. As $\mathcal{O}$ already contains all integral elements of $\left(\frac{u,\pi}{K}\right)$, we conclude that it must be the unique maximal order of $\left(\frac{u,\pi}{K}\right)$.

Theorem 5.16. *If* K *is a* p*-adic field and* A *is the unique quaternion division algebra over* K*, then the set of all integral elements of* A *is the unique maximal order of* A*.*

5.4. **A preview: type numbers.** Much of the discussion in Section 5.2 was focused on conjugacy classes of maximal orders in quaternion matrix algebras, while in Section 5.3 we saw that the unique quaternion division algebra over a p-adic field contains a unique maximal order. In this section we let R be a Dedekind domain with quotient field K and A a quaternion algebra over K.

Suppose that $\mathcal{O}_1$ and $\mathcal{O}_2$ are orders in A that are isomorphic via some isomorphism $f : \mathcal{O}_1 \to \mathcal{O}_2$. Via extension of scalars the map f induces an isomorphism $\hat{f} : \mathcal{O}_1 \otimes_R K \to \mathcal{O}_2 \otimes_R K$ for which $\hat{f}(x) = f(x)$ for all $x \in \mathcal{O}_1$. Because $\mathcal{O}_1$ and $\mathcal{O}_2$ are orders in A, $\mathcal{O}_1 \otimes_R K \cong A \cong \mathcal{O}_2 \otimes_R K$. Therefore $\hat{f}$ is an automorphism of A and is therefore given by conjugation by an element $a \in A^*$ by the Skolem-Noether theorem. In particular $\mathcal{O}_2 = a\mathcal{O}_1 a^{-1}$. The moral is that in a quaternion algebra two orders are isomorphic if and only if they are conjugate.

Definition 5.17. *The* **type number** *of a quaternion algebra is the number of conjugacy classes of maximal orders.*

As was mentioned earlier, the two situations we are most interested in are when K is a number field or a p-adic field. In the latter case the type number is always one. Indeed, there are only two isomorphism classes of quaternion algebras over a non-archimedean local field ($\mathrm{M}_2(K)$ or the unique quaternion division algebra) and we have shown in Section 5.2 that $\mathrm{M}_2(K)$ has type number one. In Section 5.3 we saw that the unique quaternion division algebra over a p-adic field has an even stronger property: the set of all R-integral elements has the structure of an order which by necessity is the unique maximal order. It therefore has type number one as well.

The situation is much more nuanced for quaternion algebras over number fields and its calculation is in some ways reminiscent of the study of the class number. In this case the type number is always finite (which is not a priori obvious), though it can be arbitrarily large. When A is unramified at an archimedean prime of K we will see that the type number is actually always a power of 2.

The type number of a quaternion algebra over a number field plays a crucial role in Vignéras construction of isospectral hyperbolic 3-manifolds, as well as in applications to other fields like modular forms. We will study this quantity in some depth in later lectures. For now we merely state a general result.

Theorem 5.18. *If K is a local or global field and A is a quaternion algebra over K then the type number of A is finite.*

Type numbers are very easy to compute using Magma.

Example 5.19. Let $k = \mathbb{Q}(\sqrt{-10})$ and consider the quaternion algebra $A = \left(\frac{-1,-3}{k}\right)$. We will show that the type number of A is 2 and compute generating sets for representatives of the two conjugacy classes of maximal orders of A (considered as modules over $\mathcal{O}_k$).

```
> k<t>:=QuadraticField(-10);
> t^2;
-10
> A<i,j,ij>:=QuaternionAlgebra<k|-1,-3>;
> C:=ConjugacyClasses(MaximalOrder(A));
```

```
> #C;
2
IsConjugate(C[1],C[2]);
false
> Generators(C[1]);
[ 1, i, 1/2*i + 1/2*j, 1/2 + 1/2*t*i + 1/6*t*j + 1/6*ij ]
> Generators(C[2]);
[ 1, 2*i, 3*t*i, 1 + 1/2*i + 1/2*j, 1/2*(t + 2) + 1/4*(t + 2)*i + 1/4*(t
+ 2)*j, 1/2*(t + 1) + 1/4*(t + 4)*i - 1/12*t*j + 1/6*ij ]
```

Example 5.20. Corollary 5.13 showed that if $\mathcal{O}_k$ is a PID then the type number of $\mathrm{M}_2(k)$ is one. The ring of integers $\mathcal{O}_k$ is a PID if and only if k has class number one, so the corollary proves that the type number of $\mathrm{M}_2(k)$ coincides with the class number h of k when $h = 1$. The discussion following the proof of Corollary 5.13 provides intuition for something that we will provide a short proof of once we have more techniques at our disposal: the type number of $\mathrm{M}_2(k)$ always coincides with the cardinality of the ideal class group of k modulo squares (for any choice of number field k). In particular if k has class number 2 then the type number of $\mathrm{M}_2(k)$ is 2 as well.

As in the previous example, let $k = \mathbb{Q}(\sqrt{-10})$. This time however, let $A = \mathrm{M}_2(k)$, or in terms of Hilbert symbols let $A = \left(\frac{1,1}{k}\right)$. In this case k has class number 2. We will use Magma to show that A has type number 2, as was asserted above.

```
> k<t>:=QuadraticField(-10);
> ClassNumber(k);
2
> A<i,j,ij>:=QuaternionAlgebra<k|1,1>;
> #ConjugacyClasses(MaximalOrder(A));
2
```

6. Arithmetic Kleinian groups

In this section we will construct discrete subgroups of $\mathrm{PSL}_2(\mathbb{C})$ from orders in quaternion algebras and relate the geometric properties of the resulting Kleinian groups to algebraic properties of the associated quaternion algebras. This will allow us to define what it means for a Kleinian group to be arithmetic. We will conclude by showing that the invariant trace field and quaternion algebra are complete commensurability invariants of arithmetic Kleinian groups.

Throughout this section we will employ the following notation. Let k be a number field with ring of integers $\mathcal{O}_k$ and A be a quaternion algebra over k. An *order* $\mathcal{O}$ *of* A will always mean that $\mathcal{O}$ is an $\mathcal{O}_k$-order of A. If B is a subring of A then we will denote by B^1 the multiplicative subgroup of B^* generated by elements having reduced norm 1.

Finally, we recall that $\mathrm{Ram}(A)$ (respectively $\mathrm{Ram}_f(A)$ or $\mathrm{Ram}_\infty(A)$) denotes the set of all places of k (respectively finite or infinite) which ramify in A.

6.1. Discrete groups from orders in quaternion algebras.

Let k be a number field of degree n with a unique complex place ν. Recall that this just means that of the n embeddings $\sigma : k \hookrightarrow \mathbb{C}$, the image $\sigma(k)$ of k will lie in $\mathbb{R}$ for precisely $n-2$ embeddings. The other two embeddings will be given by ν and $\overline{\nu}$, the complex conjugate of ν. Let S_∞ denote the set of archimedean places of k.

Now suppose that A is a quaternion algebra over k which is ramified at all real places of k. Recalling that there is always an isomorphism

$$A \otimes_{\mathbb{Q}} \mathbb{R} \cong \bigoplus_{v \in S_\infty} A \otimes_k k_v,$$

we deduce that

$$A \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{H}^{n-2} \times \mathrm{M}_2(\mathbb{C}). \tag{4}$$

Let $\psi : A \hookrightarrow \mathrm{M}_2(\mathbb{C})$ denote the composition of the natural embedding $A \hookrightarrow A \otimes_{\mathbb{Q}} \mathbb{R}$ with the isomorphism in (4) and the projection map from $\mathbb{H}^{n-2} \times \mathrm{M}_2(\mathbb{C})$ onto $\mathrm{M}_2(\mathbb{C})$.

Theorem 6.1. *Let $\mathcal{O}$ be a maximal order of A and $\Gamma_{\mathcal{O}} = P\psi(\mathcal{O}^1) \subset \mathrm{PSL}_2(\mathbb{C})$.*

(1) $\Gamma_{\mathcal{O}}$ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$.

(2) The volume of $\mathbf{H}^3/\Gamma_{\mathcal{O}}$ is given by

$$\mathrm{Vol}(\mathbf{H}^3/\Gamma_{\mathcal{O}}) = \frac{d_k^{3/2}\zeta_k(2)}{(4\pi^2)^{[k:\mathbb{Q}]-1}} \cdot \left(\prod_{\mathfrak{p} \in \mathrm{Ram}_f(A)} (N(\mathfrak{p}) - 1) \right),$$

where d_k is the absolute value of the discriminant of k and $\zeta_k(2)$ is the Dedekind zeta function of k evaluated at $s = 2$.

Proof. For a proof that $\Gamma_{\mathcal{O}}$ is discrete, see [36, Chapitre IV, Theoreme 1.1]. The formula for the covolume of $\Gamma_{\mathcal{O}}$ is due to Borel [4, Section 7.3]. □

We may now, at long last, define what it means for a Kleinian group to be arithmetic.

Definition 6.2. *Let k be a number field with a unique complex place, A a quaternion algebra over k which is ramified at all real places of k and $\mathcal{O}$ a maximal order of A.*

A subgroup of $\mathrm{PSL}_2(\mathbb{C})$ *is an* **arithmetic Kleinian group** *if it is commensurable with* $\Gamma_{\mathcal{O}}$ *for some triple* $(k, A, \mathcal{O})$ *as above.*

The groups $\Gamma_{\mathcal{O}}$ will be important enough in our discussion of arithmetic Kleinian groups that it will be useful to provide them with a notation-free name. Henceforth we will refer to groups of the form $\Gamma_{\mathcal{O}}$ as **arithmetic Kleinian groups of the simplest type**. Thus a subgroup of $\mathrm{PSL}_2(\mathbb{C})$ is an arithmetic Kleinian group if and only if it is commensurable to an arithmetic Kleinian group of the simplest type.

Recall that Wedderburn's theorem tells us that if a quaternion algebra over k is not isomorphic to $\mathrm{M}_2(k)$ then it is a division algebra. In the context of quaternion algebras A ramified at all real places of k, this means that A will fail to be a division algebra if and only if k has no real places (i.e., $k = \mathbb{Q}(\sqrt{-d})$ for some positive square-free integer d) and $A \cong \mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$.

Example 6.3. Consider an imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ with ring of integers $\mathcal{O}_d$. In Lemma 5.10 we saw that $\mathrm{M}_2(\mathcal{O}_d)$ is a maximal order in the quaternion algebra $\mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$. The group $\mathrm{PSL}_2(\mathcal{O}_d)$ is called a **Bianchi group**. Every Bianchi group contains the parabolic isometry $z \mapsto z+1$ of $\hat{\mathbb{C}}$ and is thus non-cocompact. In fact it is well-known that the number of cusps of the Bianchi group associated to $\mathbb{Q}(\sqrt{-d})$ is equal to the ideal class number of $\mathbb{Q}(\sqrt{-d})$. According to Theorem 6.1, $\mathrm{PSL}_2(\mathcal{O}_3)$ is the Bianchi group of smallest covolume. Using Magma it is easy to compute the covolumes of Bianchi groups.

TABLE 1. Volumes of small Bianchi orbifolds

d	$\mathrm{Vol}(\mathbf{H}^3/\mathrm{PSL}_2(\mathcal{O}_d))$
1	0.30532186472574...
2	1.00384100334120...
3	0.16915693440160...
5	4.20396925947605...
6	5.18217289781959...
7	0.88891492781635...
10	9.81811844389802...
11	1.38260830790264...
13	13.9979614019778...
14	20.3513407500735...

Given a positive square-free integer d, the volume $\mathrm{Vol}(\mathbf{H}^3/\mathrm{PSL}_2(\mathcal{O}_d))$ can be computed in Magma with the following commands:

```
> RR := RealField();
> pi := Pi(RR);
> R<x>:=PolynomialRing(Rationals());
> k:=NumberField(x^2+d);
> Dk:=Abs(Discriminant(Integers(k)));
> Zeta:=Evaluate(LSeries(k),2);
```

```
> Dk^(3/2)*Zeta/(4*pi^2);
```

While it is of course not necessary to use a computer algebra system to compute the discriminant of a quadratic field, the above code can easily be modified in order to compute the volume of arbitrary arithmetic Kleinian groups of the form $\Gamma_{\mathcal{O}}$. One simply needs to replace x^2+d with the defining polynomial of k and remember to account for the

$$\left(\prod_{\mathfrak{p}\in \mathrm{Ram}_f(A)} (N(\mathfrak{p})-1)\right)$$

term appearing in Theorem 6.1, as this term is trivial in the case that $A \cong \mathrm{M}_2(k)$.

The following theorem relates the topology of an arithmetic hyperbolic 3-manifold $\mathbf{H}^3/\Gamma$ to the structure of an associated quaternion algebra.

Theorem 6.4. *Let $M = \mathbf{H}^3/\Gamma$ be an arithmetic hyperbolic 3-manifold and suppose that Γ is commensurable with $\Gamma_{\mathcal{O}}$, where $\mathcal{O}$ is a maximal order in a quaternion algebra A over a number field k. The following are equivalent:*

(1) M is non-compact.
(2) k is an imaginary quadratic field and $A \cong \mathrm{M}_2(k)$.
(3) Γ is commensurable in the wide sense with a Bianchi group.

Proof. If Γ is not cocompact then neither is $\Gamma_{\mathcal{O}}$. Thus $\Gamma_{\mathcal{O}}$ contains a parabolic element γ. As the element $\gamma - \mathrm{Id}$ is not invertible we may conclude that A is not a division algebra. By Wedderburn's theorem, $A \cong M_2(k)$. We have already seen that a maximal order of $\mathrm{M}_2(k)$ will yield an arithmetic Kleinian group only if k is imaginary quadratic. Therefore (1) implies (2). That (2) implies (3) follows from the definition of a Bianchi group and the fact that maximal orders in the same quaternion algebra will always yield arithmetic Kleinian groups which are commensurable. To prove that (3) implies (1) we note that all Bianchi groups contain parabolic elements, hence Γ will as well if Γ is commensurable in the wide sense to a Bianchi group. □

As a corollary of Theorem 6.4 we obtain the following.

Corollary 6.5. *Let $M = \mathbf{H}^3/\Gamma$ be an arithmetic hyperbolic 3-manifold and suppose that Γ is commensurable with $\Gamma_{\mathcal{O}}$, where $\mathcal{O}$ is a maximal order in a quaternion algebra A over a number field k. The manifold M is compact if and only if A is a division algebra.*

Theorem 6.6. *Let Γ be a cocompact arithmetic Kleinian group of the simplest type. Then the covolume of Γ is greater than $0.888\ldots$ This volume is achieved by $\Gamma_{\mathcal{O}}$ where $\mathcal{O}$ is a maximal order in the quaternion algebra over $\mathbb{Q}(\sqrt{-7})$ ramified at the two primes of norm 2.*

Proof. In light of Theorem 6.1(2) it suffices to show that if $\Gamma_{\mathcal{O}}$ arises from some (k, A) with $k \neq \mathbb{Q}(\sqrt{-7})$ then $\mathrm{Vol}(\mathbf{H}^3/\Gamma_{\mathcal{O}}) \geq 0.889$. This is sufficient because the choice of division algebra A which will yield the smallest volume $\Gamma_{\mathcal{O}}$ is clearly the one which is ramified at the two primes of norm 2. (Since ramification at any other prime would contribute to the volume of $\mathbf{H}^3/\Gamma_{\mathcal{O}}$ by Theorem 6.1(2).)

We will first show that if $\mathrm{Vol}(\mathbf{H}^3/\Gamma_{\mathcal{O}}) \leq 0.889$ then $[k:\mathbb{Q}] = 2$. Let $\zeta_k(s)$ denote the Dedekind zeta function of k. The Euler product expansion

$$\zeta_k(s) = \prod_{\mathfrak{p}} \frac{1}{1 - N(\mathfrak{p})^{-s}},$$

where the product is taken over prime ideals of $\mathcal{O}_k$, implies that $\zeta_k(2) \geq 1$. Every prime ideal $\mathfrak{p}$ of $\mathcal{O}_k$ satisfies $N(\mathfrak{p}) \geq 2$, hence

$$\left(\prod_{\mathfrak{p} \in \mathrm{Ram}_f(A)} (N(\mathfrak{p}) - 1) \right) \geq 1$$

as well. Theorem 6.1(2) now implies that

$$\mathrm{Vol}(\mathbf{H}^3/\Gamma_{\mathcal{O}}) \geq \frac{d_k^{3/2}}{4\pi^2}.$$

We are therefore assuming that

$$0.889 \geq \mathrm{Vol}(\mathbf{H}^3/\Gamma_{\mathcal{O}}) \geq \frac{d_k^{3/2}}{4\pi^2},$$

whence $10.720 \geq d_k$. There are only four number fields with absolute value of discriminant less than 11 which also have a unique complex place and all of these fields are imaginary quadratic. Note that claims like this can be proven in at least two different ways. On the one hand this is the type of claim that can be proven using Odlyzko discriminant bounds [28]. When the discriminant is very small however, it is easier to use (provably complete in the appropriate range) databases of number fields of small discriminant (c.f. [10]).

Observe that if $\Gamma_{\mathcal{O}}$ is a cocompact arithmetic Kleinian group of the simplest type which arises from (k, A) then A is a division algebra by Corollary 6.5. In particular this means that if k is an imaginary quadratic field then A is ramified at at least two finite primes. It follows from the volume formula that over each of the four imaginary quadratic fields with absolute value of discriminant less than 11, the cocompact arithmetic Kleinian group of the simplest type will be ramified at the two primes of smallest norm in the field. We now compute with Magma the corresponding volume of $\mathbf{H}^3/\Gamma_{\mathcal{O}}$. This yields the data in Table 2 and concludes our proof. □

Table 2. Small cocompact arithmetic Kleinian groups of the simplest type

$k = \mathbb{Q}(\sqrt{-d})$	Norms of two smallest primes of $\mathcal{O}_k$	$\mathrm{Vol}(\mathbf{H}^3/\Gamma_{\mathcal{O}})$
$\mathbb{Q}(\sqrt{-1})$	2,5	1.221287458902...
$\mathbb{Q}(\sqrt{-2})$	2,3	2.007682006682...
$\mathbb{Q}(\sqrt{-3})$	3,4	1.014941606409...
$\mathbb{Q}(\sqrt{-7})$	2,2	0.888914927816...

6.2. **A second characterization of arithmeticity.** In Section 6.1 we defined arithmetic Kleinian groups. Our definition was in terms of commensurability; namely, a Kleinian group Γ is arithmetic if and only if it is commensurable with a group of the form $\Gamma_{\mathcal{O}} = P\psi(\mathcal{O}^1)$, where $\mathcal{O}$ is a maximal order in a quaternion algebra A (ramified at all real places) over a number field k having a unique complex place. Here ψ denotes the natural map $A \hookrightarrow A \otimes_k \mathbb{C} \to \mathrm{M}_2(\mathbb{C})$.

We begin this section by identifying the invariant trace fields $k\Gamma$ and quaternion algebras $A\Gamma$ of an arithmetic Kleinian group of the simplest type $\Gamma_{\mathcal{O}}$. Recall that these were defined as $k\Gamma = \mathbb{Q}(\operatorname{tr} \Gamma^{(2)})$ and

$$A\Gamma = \left\{ \sum a_i \gamma_i : a_i \in k\Gamma, \gamma_i \in \Gamma^{(2)} \right\},$$

where $\Gamma^{(2)}$ is the subgroup of Γ generated by squares.

Theorem 6.7. *Let Γ be an arithmetic Kleinian group which is commensurable to $\Gamma_{\mathcal{O}} = P\psi(\mathcal{O}^1)$, where $\mathcal{O}$ is a maximal order in a quaternion algebra A defined over the field k and satisfies the conditions above. Then $k\Gamma = k$ and $A\Gamma = \psi(A)$.*

Proof. Because $k\Gamma$ and $A\Gamma$ are invariants of the commensurability class of Γ (Theorem 4.14 and Corollary 4.15) it suffices to prove the theorem for $\Gamma = \Gamma_{\mathcal{O}}$. We begin by observing that $\mathbb{Q}(\operatorname{tr} \Gamma_{\mathcal{O}}^{(2)}) \subset k$, as every element of $\mathcal{O}$ is integral and thus has a reduced trace lying in $\mathcal{O}_k \subset k$. On the one hand Corollary 4.19 implies that the field $\mathbb{Q}(\operatorname{tr} \Gamma_{\mathcal{O}}^{(2)})$ is not totally real. On the other hand every subfield of k is totally real. (This is true more generally: every subfield of a number field having a unique complex place is totally real.) We conclude that $k = \mathbb{Q}(\operatorname{tr} \Gamma_{\mathcal{O}}^{(2)}) = k\Gamma_{\mathcal{O}}$.

It remains to show that $A\Gamma = \psi(A)$. Consider the quaternion algebra $\psi(A)$. In light of the previous paragraph it is a quaternion algebra over $k\Gamma_{\mathcal{O}}$ which obviously contains $A_0\Gamma_{\mathcal{O}}^{(2)} = A\Gamma_{\mathcal{O}}$ (since $A_0\Gamma_{\mathcal{O}}^{(2)}$ is generated over $k\Gamma$ by elements of the form γ^2 for $\gamma \in \mathcal{O}^1$ and these are by definition elements of $\psi(A)$). Because both algebras are quaternion algebras over $k\Gamma_{\mathcal{O}}$, a dimension count implies that $A\Gamma_{\mathcal{O}} = \psi(A)$. □

The following is an immediate corollary of Theorem 6.7.

Corollary 6.8. *Two arithmetic Kleinian groups of the simplest type $\Gamma_{\mathcal{O}}$ and $\Gamma_{\mathcal{O}'}$ are commensurable if and only if $\mathcal{O}$ and $\mathcal{O}'$ are maximal orders in the same quaternion algebra, up to isomorphism. In particular distinct Bianchi groups are never commensurable.*

We have now shown that the invariant trace field of an arithmetic Kleinian group is a number field possessing a unique complex place and that the invariant quaternion algebra is a quaternion algebra over this number field which is ramified at all real places. We make an additional observation about arithmetic Kleinian groups. Suppose that Γ is such a group and $\gamma \in \Gamma$. There exists an integer $n > 0$ such that $\gamma^n \in \Gamma_{\mathcal{O}}$ for some $\mathcal{O}$. It follows that $\operatorname{tr} \gamma^n \in \mathcal{O}_k$, hence $\operatorname{tr} \gamma$ satisfies a monic polynomial with coefficients in $\mathcal{O}_k$. Therefore $\operatorname{tr} \gamma$ is an algebraic integer and $\operatorname{tr} \Gamma \subset \mathcal{O}_k$.

The following result shows that arithmetic Kleinian groups are in fact characterized by these properties.

Theorem 6.9. *A finite-covolume Kleinian group Γ is arithmetic if and only if the following conditions hold.*

(1) *The field $k\Gamma$ is a number field with a unique complex place.*
(2) *The algebra $A\Gamma$ is ramified at all real places of $k\Gamma$.*
(3) *The element $\operatorname{tr}\gamma$ is an algebraic integer for all $\gamma \in \Gamma$.*

Proof. The main ideas of the proof are exactly as in the proof of Theorem 4.3 and rely on showing that the set

$$\mathcal{O}\Gamma = \left\{ \sum a_i \gamma_i : a_i \in \mathcal{O}_{k\Gamma}, \gamma_i \in \Gamma^{(2)} \right\}$$

is an order of $A\Gamma$. For a full proof we refer the reader to [24, Theorem 8.3.2]. □

Our discussion of the Weeks manifold (in particular Theorem 4.23) now shows the following.

Corollary 6.10. *The Weeks manifold is arithmetic.*

6.3. **Complete commensurability invariants.** Recall that we have shown (Theorem 4.14 and Corollary 4.15) that the invariant trace field and quaternion algebra are commensurability class invariants of finite-covolume Kleinian groups. It is not in general the case that these are complete commensurability invariants however. That is, it is not in general true that non-commensurable Kleinian groups will have non-isomorphic invariant quaternion algebras. The following result shows that the invariant trace field and quaternion algebra are complete commensurability class invariants when restricted to the class of arithmetic Kleinian groups.

Theorem 6.11. *Let Γ_1, Γ_2 be arithmetic Kleinian groups. Then Γ_1 and Γ_2 are commensurable in the wide sense if and only if $k\Gamma_1 = k\Gamma_2$ and there exists a $k\Gamma_1$-isomorphism $\phi : A\Gamma_1 \to A\Gamma_2$.*

Proof. We begin by assuming that Γ_1 and Γ_2 are commensurable in the wide sense. Let $g \in \mathrm{SL}_2(\mathbb{C})$ be such that $g\Gamma_1 g^{-1}$ and Γ_2 are directly commensurable. As traces are invariant under conjugation and commensurable groups have identical invariant trace fields, we see that $k\Gamma_1 = k\left(g\Gamma_1 g^{-1}\right) = k\Gamma_2$. The map

$$\phi : A\Gamma_1 \to A\Gamma_2$$

given by

$$\phi\left(\sum a_i\gamma_i\right) = \sum a_i g\gamma_i g^{-1}$$

is moreover the required $k\Gamma_1$-isomorphism.

Now suppose that conversely, $k\Gamma_1 = k\Gamma_2$ and that $\phi : A\Gamma_1 \to A\Gamma_2$ is a $k\Gamma_1$-isomorphism. The Skolem-Noether theorem implies that there exists an element $g \in A\Gamma_2^*$ such that $\phi(x) = gxg^{-1}$ for all $x \in A\Gamma_1$. Then $\phi(\mathcal{O}\Gamma_1)$ is an order in $A\Gamma_2$. As Γ_i is commensurable with $\mathcal{O}\Gamma_i^1$ for $i = 1, 2$ we conclude that $g\Gamma_1 g^{-1}$ is commensurable with Γ_2. □

The following is an immediate application of Theorem 6.11.

Corollary 6.12. *The set of invariant trace fields of arithmetic Kleinian groups coincides exactly with the set of number fields having a unique complex place. For every such number field k there are infinitely many commensurability classes of arithmetic Kleinian groups having k as their invariant trace field.*

6.4. **A third characterization of arithmeticity.** Thus far we have seen two definitions of arithmeticity. The first said that a Kleinian group of finite covolume is arithmetic if it is commensurable to an arithmetic Kleinian group of the simplest type (i.e., one of the form $\Gamma_{\mathcal{O}}$). This definition is of a constructive nature and is most useful for showing that certain explicitly constructed Kleinian groups are arithmetic. For instance we showed that the Bianchi groups were arithmetic in this manner. The second definition (Theorem 6.9) said that a Kleinian group of finite covolume is arithmetic if it satisfies three properties, two dealing with the structure of its invariant trace field and quaternion algebra and the third asserting that all elements have traces that are algebraic integers. This definition tends to be most useful in proving the arithmeticity of Kleinian groups whose presentations are known. For instance we showed that the Weeks manifold is arithmetic in this manner. In this brief section we will give a third definition of arithmeticity; this one of a geometric nature.

Let Γ be a finite covolume subgroup of $\mathrm{PSL}_2(\mathbb{C})$. The **commensurator** $\mathrm{Comm}(\Gamma)$ of Γ is defined by

$$\mathrm{Comm}(\Gamma) = \{g \in \mathrm{PSL}_2(\mathbb{C}) : g\Gamma g^{-1} \text{ is commensurable with } \Gamma\}.$$

Theorem 6.13. *If Γ is an arithmetic Kleinian group then* $\mathrm{Comm}(\Gamma) = P(A\Gamma^*)$.

Proof. Commensurable groups have the same commensurator, hence we may assume that $\Gamma = \Gamma_{\mathcal{O}}$ for some maximal order $\mathcal{O}$ (subject to the usual constraints on k and A). If $g \in A\Gamma^*$ then $g\mathcal{O}g^{-1}$ is also maximal order of $A\Gamma$ and the groups $\Gamma_{\mathcal{O}}$ and $\Gamma_{g\mathcal{O}g^{-1}} = g\Gamma_{\mathcal{O}}g^{-1}$ are commensurable. Therefore $g \in \mathrm{Comm}(\Gamma)$, and as g was arbitrary we have $P(A\Gamma^*) \subset \mathrm{Comm}(\Gamma)$.

We now prove the reverse inclusion. Suppose that $g \in \mathrm{Comm}(\Gamma)$ and let Γ_0 be a non-elementary subgroup of $\Gamma^{(2)}$ such that we have the inclusion $\Gamma_0, g\Gamma_0 g^{-1} \subset \Gamma^{(2)}$. Conjugation by g induces an automorphism of $A\Gamma$ which by Skolem-Noether is given by conjugation by an element $x \in A\Gamma^*$. Observing that

$$x\gamma x^{-1} = g\gamma g^{-1}$$

for all $\gamma \in \Gamma^{(2)}$, we deduce that $gx^{-1} \in Z(A\Gamma^*) = k\Gamma$. It follows that $g \in P(A\Gamma^*)$, concluding the proof. □

Theorem 6.13 shows that if Γ is an arithmetic Kleinian group then Γ has infinite index in its commensurator. Margulis' celebrated arithmeticity theorem, on the other hand, provides a converse to this result and provides us with our third characterization of arithmeticity.

Theorem 6.14 (Margulis)**.** *A finite covolume Kleinian group is arithmetic if and only if it has infinite index inside of its commensurator.*

7. Isospectral arithmetic hyperbolic 3-manifolds are commensurable

In this section we prove that compact arithmetic hyperbolic 3-manifolds which are isospectral are necessarily commensurable, a result originally due to Reid [31].

Let M be a compact arithmetic hyperbolic 3-manifold and $\mathcal{E}(M)$ be the multiset of eigenvalues of the Laplace-Beltrami operator acting on $L^2(M)$. We call $\mathcal{E}(M)$ the **Laplace eigenvalue spectrum** of M. It is known that $\mathcal{E}(M)$ is a discrete, infinite subset of the positive real numbers. If M and N are compact arithmetic hyperbolic 3-manifolds for which $\mathcal{E}(M) = \mathcal{E}(N)$, then we say that M and N are **isospectral**.

We now define another geometric invariant of M. Let $\mathcal{L}(M)$ denote the multiset of lengths of closed geodesics of M. We call $\mathcal{L}(M)$ the **length spectrum** of M. As was the case with the Laplace eigenvalue spectrum of M, the length spectrum $\mathcal{L}(M)$ is a discrete subset of the positive real numbers. If M and N are compact arithmetic hyperbolic 3-manifolds for which $\mathcal{L}(M) = \mathcal{L}(N)$, then we say that M and N are **length-isospectral**.

Although the multiplicities of elements in $\mathcal{L}(M)$ contain an enormous amount of information, we will oftentimes only need knowledge of $\mathcal{L}(M)$ *as a set*. For this reason we define the **weak length spectrum** $L(M)$ of M to be the set of lengths of closed geodesics of M (i.e., without multiplicities).

In summary we have just defined three geometric invariants of M:

$\mathcal{E}(M) = \{\lambda : \lambda \text{ is an eigenvalue of the Laplace-Beltrami operator of } M\}$,
$\mathcal{L}(M) = \{\ell : \ell \text{ is the length of a closed geodesic on } M\}$,
$L(M) = \{\ell : \ell \in \mathcal{L}(M)\}$.

We stress that whereas $\mathcal{E}(M)$ and $\mathcal{L}(M)$ are multisets, $L(M)$ is a set and does not contain elements with multiplicity greater than one.

The relationship between these invariants are given by the following theorems:

Theorem 7.1 (Kelmer [21]). *Let M and N be compact hyperbolic 3-manifolds. If $\mathcal{L}(M) = \mathcal{L}(N)$ then $\mathcal{E}(M) = \mathcal{E}(N)$.*

On the other hand, an application of the Duistermaat-Guillemin trace formula [14] (see also [29, Theorem 10.1]) provides a partial converse.

Theorem 7.2 (Duistermaat-Guillemin). *Let M and N be compact hyperbolic 3-manifolds. If $\mathcal{E}(M) = \mathcal{E}(N)$ then $L(M) = L(N)$.*

We note that the Duistermaat-Guillemin trace formula is much more general than the manner in which we have stated it and applies to compact locally symmetric spaces. It is unknown whether or not the Laplace eigenvalue spectrum of a compact hyperbolic 3-manifold determines the length spectrum with multiplicities. There *are* examples of compact Riemannian manifolds with the same eigenvalue spectra but different length spectra, though such examples are not hyperbolic.

7.1. Fields generated by eigenvalues.

We have already seen that the field generated over $\mathbb{Q}(\operatorname{tr}\Gamma)$ by the eigenvalues of a loxodromic element of a finite-covolume Kleinian group has degree at most 2. We now study these fields in more detail.

Lemma 7.3. *Let Γ be a Kleinian group with finite-covolume and assume that $k\Gamma = \mathbb{Q}(\operatorname{tr}\Gamma)$. For all nontrivial elements $\gamma \in \Gamma$ the field $k\Gamma(\lambda_\gamma)$ embeds into $A_0\Gamma$.*

Proof. If the characteristic polynomial of γ splits over $k\Gamma$ then $\lambda_\gamma \in k\Gamma$ and the lemma is obvious. Suppose therefore that the characteristic polynomial is an irreducible quadratic polynomial and define a subalgebra B of $A_0\Gamma$ by

$$B = \{a + b\gamma : a, b \in k\Gamma\}.$$

It is clear that B is a commutative subalgebra of $A_0\Gamma$ which properly contains $k\Gamma$. It follows that B is a quadratic field extension of $k\Gamma$ and that $k\Gamma(\lambda_\gamma)$ embeds into $A_0\Gamma$ via the map $\lambda_\gamma \mapsto \gamma$. □

Note that we have already seen many examples of groups which satisfy the hypothesis of Lemma 7.3. For instance, if Γ is a finite-index subgroup of $\Gamma_{\mathcal{O}}$ then $k = k\Gamma = \mathbb{Q}(\operatorname{tr}\Gamma)$ and $A_0\Gamma = A\Gamma = A$.

Because we will be making repeated use of embeddings of quadratic extensions into quaternion algebras we recall the important consequence of the Albert-Brauer-Hasse-Noether theorem.

Theorem 7.4 (Albert-Brauer-Hasse-Noether)**.** *Let k be a number field and A a quaternion algebra over k. Let L be a quadratic field extension of k. The following are equivalent.*

(1) L embeds into A.
(2) $A \otimes_k L \cong \mathrm{M}_2(L)$.
(3) No place of k that ramifies in A splits in L/k.

Corollary 7.5. *Let Γ be a Kleinian group with finite-covolume and assume that $k\Gamma = \mathbb{Q}(\operatorname{tr}\Gamma)$. If $\gamma \in \Gamma$ is nontrivial then $A_0\Gamma$ admits an embedding of $k\Gamma(\lambda_\gamma)$.*

Proof. If $A_0\Gamma$ is a division algebra then this follows from Lemma 7.3 and Theorem 7.4. If $A_0\Gamma$ is not a division algebra then $A_0\Gamma \cong \mathrm{M}_2(k\Gamma)$ and therefore admits an embedding of every quadratic extension of $k\Gamma$. □

Lemma 7.6. *Let $\Gamma_{\mathcal{O}}$ be an arithmetic Kleinian group of the simplest type and Γ be a finite index subgroup of $\Gamma_{\mathcal{O}}$. If $\gamma \in \Gamma$ is loxodromic then $[k(\lambda_\gamma) : k] = 2$.*

Proof. We have already seen that this holds when A is a division algebra. Assume therefore that $A = \mathrm{M}_2(k)$. Because A must ramify at all real places of k it follows that $k = \mathbb{Q}(\sqrt{-d})$ for some square-free integer $d > 0$. As $\operatorname{tr}\gamma = \lambda_\gamma + \lambda_\gamma^{-1}$ is an algebraic integer, λ_γ is a unit. If $[k(\lambda_\gamma) : k] = 1$ then $\lambda_\gamma \in k$. The only units in $\mathbb{Q}(\sqrt{-d})$ are roots of unity however, which contradicts the fact that γ is loxodromic. □

We now prove an important theorem that will allow us to construct loxodromic elements from quadratic extensions of quaternion algebras.

Theorem 7.7. *Let k be a number field with a unique complex place and A a quaternion algebra over k which is ramified at all real places of k. Let L be a quadratic field extension of k. There is an embedding of L into A if and only if there is a maximal order $\mathcal{O}$ of A and element $\gamma \in \mathcal{O}^1$ with infinite order such that $L = k(\lambda_\gamma)$.*

Proof. If such a γ exists then it is clear that L embeds into A via the map induced by $\gamma \mapsto \lambda_\gamma$. For the converse suppose that L embeds into A. Because k has a unique complex place, Dirichlet's Unit Theorem implies that $\mathcal{O}_L^*$ has rank over $\mathbb{Z}$ strictly greater than $\mathcal{O}_k^*$. In particular there exists an element $y \in \mathcal{O}_L^*$ such that $y^n \notin \mathcal{O}_k^*$ for any $n \geq 1$. Let σ denote the nontrivial element of $\mathrm{Gal}(L/k)$ and define $u = y/\sigma(y)$ so that $N_{L/k}(u) = 1$ and $u^n \notin \mathcal{O}_k^*$ for any $n \geq 1$. (If $y^n = t\sigma(y)^n$ for some $t \in k$ then applying σ to both sides of the equation and using the fact that σ has order 2 implies that $t = \pm 1$ and $y^n \in k$ for some n, a contradiction.) We conclude that $L = k(u^n)$ for every $n \geq 1$.

Since L embeds into A there exists a maximal order $\mathcal{O}$ of A which contains $\mathcal{O}_L$. The restriction of the reduced norm of A to L is the usual field norm $N_{L/k}$, hence $u \in \mathcal{O}^1$ and $\gamma = P(\psi(u)) \in \Gamma_{\mathcal{O}}$ has infinite order. □

Corollary 7.8. *Let Γ be a Kleinian group which is contained in an arithmetic Kleinian group $\Gamma_{\mathcal{O}}$ of the simplest type. Let L be a quadratic field extension of k. Then L embeds into A if and only if Γ contains an element γ of infinite order with $L = k(\lambda_\gamma)$.*

Proof. Let γ be as in Theorem 7.7 and m be such that $\gamma^m \in \Gamma$. Then $L = k(u^m) = k(\lambda_{\gamma^m})$. □

7.2. **The weak length spectrum determines the invariant trace field and invariant quaternion algebra.** We can now prove that the Laplace eigenvalue spectrum of a compact arithmetic hyperbolic 3-manifold determines its commensurability class. In light of Theorem 7.2 it suffices to prove the following even stronger result.

Theorem 7.9. *Let M, N be compact hyperbolic 3-manifolds of finite volume and assume that M is arithmetic. If $L(M) = L(N)$ then N is arithmetic and M and N are commensurable.*

Our proof will make use of the following lemma, the proof of which follows from Theorem 7.4 and Corollary 3.32.

Lemma 7.10. *Let k be a field with a number field with a unique complex place and A_1, A_2 quaternion algebras over k. If a quadratic field extension of k embeds into A_1 if and only if it embeds into A_2 then $A_1 \cong A_2$.*

We now prove Theorem 7.9.

Proof. Write $M = \mathbf{H}^3/\Gamma_M$ and $N = \mathbf{H}^3/\Gamma_N$. Since M and N are isospectral the formula relating the length of a geodesic to the trace of the associated loxodromic element implies that the fields $\mathbb{Q}(\operatorname{tr}\gamma^2 = (\operatorname{tr}\gamma)^2 - 2 : \gamma \in \Gamma_M)$ and $\mathbb{Q}(\operatorname{tr}\gamma^2 = (\operatorname{tr}\gamma)^2 - 2 : \gamma \in \Gamma_N)$ coincide. Denote these fields by k. Let A_M and A_N be the invariant quaternion algebras of M and N. We claim that any quadratic field extension L of k that embeds into A_M embeds into A_N and vice versa.

Suppose that L embeds into A_M. Then Theorem 7.7 and its corollary show that there is a maximal order $\mathcal{O}$ of A_M and an element $u \in \mathcal{O}^1$ such that $L = k(u)$. Since Γ_M and $\Gamma_{\mathcal{O}}$ are commensurable, $P(\psi(u^m)) \in \Gamma_M$ for some $m \geq 1$. By hypothesis there exists an element $\gamma \in \Gamma_N$ such that $\operatorname{tr}\gamma = \pm \operatorname{tr}\psi(u^m)$. Then $k(\lambda_\gamma) = k(\lambda_u) = L$ embeds into A_N as required, and the claim follows from Lemma 7.10.

We have shown that the invariant trace fields and invariant quaternion algebras of Γ_M and Γ_N coincide, hence it remains only to show that N is arithmetic as the theorem will then follow from Theorem 6.11. To show that N is arithmetic we will use Theorem 6.9. We have already seen that $k\Gamma_N$ is a number field with a unique complex place and that $A\Gamma_N$ is ramified at all real places of $k\Gamma_N$ (because these coincide with the invariant trace field and invariant quaternion algebra of M, which satisfies these properties). Let $\gamma \in \Gamma_N$. We must show that $\operatorname{tr}\gamma$ is an algebraic integer. As N is a compact hyperbolic 3-manifold γ must be loxodromic, and there exists an element $\gamma' \in \Gamma_M$ such that $\operatorname{tr}\gamma = \pm \operatorname{tr}\gamma'$. The arithmeticity of M implies that $\operatorname{tr}\gamma'$ is an algebraic integer, hence so is $\operatorname{tr}\gamma$. $\square$

8. A construction of Vignéras: Examples of isospectral hyperbolic 3-manifolds

In this section we will construct pairs of arithmetic hyperbolic 3-manifolds with the same Laplace eigenvalue spectrum. Our method is originally due Vignéras [35]. We will in fact see that the hyperbolic 3-manifolds we construct will have the same length spectrum and will even be **strongly isospectral**; that is, they will have the same eigenvalue spectrum with respect to any natural, self-adjoint elliptic differential operator, e.g., the Laplacian acting on p-forms.

8.1. **Generalities on isospectrality.** Let G be a semisimple Lie group and Γ a discrete cocompact subgroup of G. Denote by $L^2(\Gamma\backslash G)$ the space of square-integrable functions on $\Gamma\backslash G$ with respect to the Haar measure induced from G and by $C_c(G)$ the space of infinitely differentiable, complex valued, compactly-supported functions on G. We define a unitary operator R_Γ of G in $L^2(\Gamma\backslash G)$ by

$$(R_\Gamma(g)f)(x) = f(xg)$$

where $f \in L^2(\Gamma\backslash G), x \in \Gamma\backslash G$, and $g \in G$. If Γ' is another discrete, cocompact subgroup of G then we say that Γ and Γ' are **representation equivalent** if there exists a unitary isomorphism $T : L^2(\Gamma\backslash G) \to L^2(\Gamma'\backslash G)$ for which

$$T(R_\Gamma(g)f) = R_{\Gamma'}(g)T(f)$$

for all $g \in G$ and $f \in L^2(\Gamma\backslash G)$.

It is well-known that representation equivalence implies isospectrality with respect to the Laplace spectrum. In fact, it is a theorem of DeTurck and Gordon [12] that representation equivalence implies strong isospectrality.

Theorem 8.1 (DeTurck and Gordon)**.** *Let G be a Lie group which acts on a Riemannian manifold M by isometries. Suppose that $\Gamma, \Gamma' \leq G$ act properly discontinuously on M. If Γ and Γ' are representation equivalent then $\Gamma\backslash M$ and $\Gamma'\backslash M$ are strongly isospectral.*

Let $\phi \in C_c(G)$ and define the operator $R_\Gamma(\phi)$ on $L^2(\Gamma\backslash G)$ by

$$(R_\Gamma(\phi)f)(x) = \int_G \phi(g)f(xg)dg.$$

This operator satisfies the Selberg Trace Formula:

Theorem 8.2 (Selberg Trace Formula)**.** *We have*

$$\operatorname{tr} R_\Gamma(\phi) = \sum_{[\gamma]\in A_\Gamma} \int_{C(\gamma,\Gamma)\backslash G} \phi(g^{-1}\gamma g)dg,$$

where A_Γ denotes the set of conjugacy classes of elements in Γ and $C(\gamma, \Gamma)$ is the centralizer in Γ of γ.

Note that R_Γ is determined by its trace. This is essentially due to Dixmier [13] and uses the fact that R_Γ decomposes as a discrete sum with finite multiplicities of irreducible unitary representations of G. The idea is as follows. Let (π_i) be a collection of irreducible unitary representations of G such that for every $\Phi \in C_c(G)$ we have

$$\sum m_i \operatorname{tr} \pi_i(\Phi) = \sum n_i \operatorname{tr} \pi_i(\Phi).$$

Suppose that there is some i for which $m_i \neq n_i$. Without loss of generality we may suppose that $m_i > 0$ and $n_i = 0$. According to Dixmier [13, Propositions 5.3.1 and 6.6.5], the representations $\sum m_i \operatorname{tr} \pi_i$ and $\sum n_i \operatorname{tr} \pi_i$ are *quasi-equivalent*, a condition that forces $n_i \neq 0$.

Define the **weight** of a conjugacy class $[\gamma]$ in Γ, for a measure on $C(\gamma, \Gamma)$, to be the volume $\operatorname{vol}(C(\gamma,\Gamma)\backslash C(\gamma, G))$. One then deduces the following from the Selberg Trace Formula.

Theorem 8.3. *If two discrete cocompact subgroups $\Gamma, \Gamma' \leq G$ have the same number of conjugacy classes of fixed weight and class in G, then Γ and Γ' are representation equivalent.*

8.2. **Spectra of arithmetic Kleinian groups of the simplest type.** We recall the usual set-up. Let k be a number field containing a unique complex place and A a quaternion division algebra over k which is ramified at all real places of k. Let $\mathcal{O}, \mathcal{O}'$ be maximal orders of A and $\Gamma_{\mathcal{O}}, \Gamma_{\mathcal{O}'}$ the associated arithmetic Kleinian groups. Corollary 6.5 shows that $\Gamma_{\mathcal{O}}, \Gamma_{\mathcal{O}'}$ are cocompact.

Because we are interested in constructed hyperbolic 3-manifolds, we need to ensure that $P\psi(A^1)$ contains no nontrivial elements of finite order. Suppose therefore that $P\psi(A^1)$ contained an element of order n. Then $\cos(\pi/n) \in k$ and $k(e^{\pi i/n})$ is a quadratic extension of k which embeds into A. There are only finitely many $n \geq 4$ for which $[k(e^{\pi i/n}) : \mathbb{Q}] = 2[k : \mathbb{Q}]$, hence by employing the Albert-Brauer-Hasse-Noether theorem (Theorem 7.4) appropriately when constructing A via the set $\operatorname{Ram}(A)$ of primes that ramify in A we may assume that $P\psi(A^1)$ is torsion-free. That $\Gamma_{\mathcal{O}}, \Gamma_{\mathcal{O}'}$ are torsion-free follows.

Given a group U and element $x \in U$, denote by $[x]_U$ the conjugacy class of x in U. The following lemma is now clear.

Lemma 8.4. *The embedding ψ of A^1 into $G = \mathrm{SL}_2(\mathbb{C})$ induces a bijection between elements $\mathcal{O}^1 \setminus \{\pm 1\}$ and $\Gamma_{\mathcal{O}} \setminus \{\pm 1\}$. Let $x \in \mathcal{O}^1 \setminus \{\pm 1\}$ so that $\gamma = \psi(x)$ is the corresponding element of $\Gamma_{\mathcal{O}} \setminus \{\pm 1\}$. The centralizer $C(\gamma, \Gamma)$ corresponds to $k(x) \cap \mathcal{O}^1$ and the conjugacy class $[\gamma]_G \cap \Gamma_{\mathcal{O}}$ corresponds to $[x]_A \cap \mathcal{O}^1$.*

Note that the field $k(x)$ is a quadratic field extension of k which embeds into A and $\Omega := k(x) \cap \mathcal{O}$ is a quadratic $\mathcal{O}_k$-order of $k(x)$ which is independent of the choice of x in $[x]_{\mathcal{O}^1}$. We will call B the **order of the conjugacy class of** x. This discussion, along with Theorems 8.1 and 8.3 and a result of Eichler [15, Theorem 2], allows us to deduce the following.

Theorem 8.5. *Suppose that $\mathcal{O}^1$ and $\mathcal{O}'^1$ have the same number of conjugacy classes of elements with a fixed reduced trace and order, then $\Gamma_{\mathcal{O}}\backslash\mathbf{H}^3$ and $\Gamma_{\mathcal{O}'}\backslash\mathbf{H}^3$ are strongly isospectral.*

We have reduced our construction of isospectral hyperbolic 3-manifolds to the study of the number of conjugacy classes of elements in a quaternion order with fixed reduced trace. In order to simplify this problem even more we will make use of the following fact, proven in [24, Theorem 12.4.5].

Theorem 8.6. *Let $\mathcal{O}$ be as above and assume that $\Gamma_{\mathcal{O}}$ contains an element of trace t_0. Then the number of conjugacy classes in $\Gamma_{\mathcal{O}}$ of elements of $\Gamma_{\mathcal{O}}$ with trace t_0 is independent of the choice of maximal order $\mathcal{O}$.*

In light of Theorems 8.5 and 8.6 it suffices to show that if Ω is a quadratic $\mathcal{O}_k$-order which embeds into $\mathcal{O}$ then Ω embeds into $\mathcal{O}'$ as well. Indeed, if $\Omega = \mathcal{O}_k[x]$ then every embedding of Ω into $\mathcal{O}$ determines (and is determined by) an element of $\mathcal{O}$ with the same characteristic polynomial as x, the image in $\mathcal{O}$ of x.

Recall from Section 5.4 that the number of isomorphism classes of maximal orders of A is called the *type number* of A. We will see in the next section that when $A \otimes_{\mathbb{Q}} \mathbb{R} \not\cong \mathbb{H}^{[k:\mathbb{Q}]}$, it is always the case that the type number is a power of two. In particular, in the case we are considering above it makes sense to speak about Ω embedding into $\frac{1}{2}$ of the isomorphism classes of maximal orders of A. (This is of course an abuse of language. It would be more correct to say that Ω embeds into representatives of $\frac{1}{2}$ of the isomorphism classes of maximal orders of A.)

The question of whether every maximal order of A admits an embedding of a fixed quadratic order Ω has a long history which goes back to the work of Chevalley in the 1930s. In 1999 Chinburg and Friedman [8] completely solved this problem and showed that the proportion of isomorphism classes of maximal orders of A which admit an embedding of Ω is equal to either $0, \frac{1}{2}$ or 1. In fact, their main theorem gives necessary and sufficient conditions for each of these proportions to occur. One of the results of their paper, which will be sufficient for our purposes, is the following.

Theorem 8.7 (Chinburg-Friedman)**.** *Let k be a number field and A a quaternion algebra over k for which $A \otimes_{\mathbb{Q}} \mathbb{R} \not\cong \mathbb{H}^{[k:\mathbb{Q}]}$. If A is ramified at a finite prime of k and Ω is a quadratic $\mathcal{O}_k$-order that embeds into a maximal order of A then every maximal order of A admits an embedding of Ω.*

We will prove Theorem 8.7 in Section 9 after discussing the type number of a quaternion algebra in greater detail. From Theorem 8.7 and the discussion above we conclude the following.

Theorem 8.8. *Let k be a number field with a unique complex place and A a quaternion division algebra over k which ramifies at all real places of k. Let $\mathcal{O}, \mathcal{O}'$ be maximal orders of A for which $\Gamma_{\mathcal{O}}$ and $\Gamma_{\mathcal{O}'}$ are torsion-free. If A ramifies at a finite prime of k then the manifolds $\Gamma_{\mathcal{O}}\backslash\mathbf{H}^3$ and $\Gamma_{\mathcal{O}'}\backslash\mathbf{H}^3$ are strongly isospectral.*

Recalling that lengths of closed geodesics on a complete orientable hyperbolic 3-manifold of finite volume correspond to conjugacy classes of loxodromic elements in the associated Kleinian group, we similarly conclude the following result from Theorems 8.6 and 8.7.

Theorem 8.9. *Let k be a number field with a unique complex place and A a quaternion division algebra over k which ramifies at all real places of k. Let $\mathcal{O}, \mathcal{O}'$ be maximal orders of A for which $\Gamma_{\mathcal{O}}$ and $\Gamma_{\mathcal{O}'}$ are torsion-free. If A ramifies at a finite prime of k then the manifolds $\Gamma_{\mathcal{O}}\backslash\mathbf{H}^3$ and $\Gamma_{\mathcal{O}'}\backslash\mathbf{H}^3$ have the same length spectra; i.e., $\mathcal{L}(\Gamma_{\mathcal{O}}\backslash\mathbf{H}^3) = \mathcal{L}(\Gamma_{\mathcal{O}'}\backslash\mathbf{H}^3)$.*

Remark 8.10. *Recall that it is a theorem of Kelmer (Theorem 7.1) that if M and N are two compact hyperbolic 3-manifolds with $\mathcal{L}(M) = \mathcal{L}(N)$ then M and N are*

Laplace-isospectral. Theorem 8.9 therefore provides a second proof that $\Gamma_{\mathcal{O}}\backslash\mathbf{H}^3$ *and* $\Gamma_{\mathcal{O}'}\backslash\mathbf{H}^3$ *are Laplace-isospectral.*

In order to make sure that the hyperbolic 3-manifolds we construct are not isometric we first note that if $\Gamma_{\mathcal{O}}\backslash\mathbf{H}^3$ and $\Gamma_{\mathcal{O}'}\backslash\mathbf{H}^3$ were isometric then there would be an element γ in $\mathrm{PGL}_2(\mathbb{C})$ for which $\Gamma_{\mathcal{O}} = \gamma\Gamma_{\mathcal{O}'}\gamma^{-1}$. The following proposition shows that this in turn proves that $\mathcal{O}$ and $\mathcal{O}'$ are conjugate in A^*. In order to obtain manifolds which are not isometric it therefore suffices to choose maximal orders which have different types; that is, which are not conjugate in A^*.

Proposition 8.11. *Let notation be as above and suppose that* $\Gamma_{\mathcal{O}} = \gamma\Gamma_{\mathcal{O}'}\gamma^{-1}$ *for some* $\gamma \in \mathrm{PGL}_2(\mathbb{C})$. *Then* $\mathcal{O}$ *and* $\mathcal{O}'$ *are conjugate in* A^*.

Proof. Let $\gamma = P(c)$ where $c \in \mathrm{GL}_2(\mathbb{C})$. Then $\psi(A) = A\Gamma_{\mathcal{O}} = A\Gamma_{\mathcal{O}'}$, hence conjugation by c induces a k-automorphism of A via

$$\sum a_i\gamma_i \mapsto \sum a_i c\gamma_i c^{-1}$$

for $a_i \in k$ and $\gamma_i \in \psi(\mathcal{O}^1)$. By the Skolem-Noether theorem this is an inner automorphism and there exists an element $a \in A^*$ such that $a\mathcal{O}^1a^{-1} = \mathcal{O}'^1$. Now consider the order $\mathcal{O}\Gamma_{\mathcal{O}}$ of $\psi(A)$ defined by

$$\mathcal{O}\psi(\mathcal{O}^1) := \left\{\sum a_i\gamma_i : a_i \in \mathcal{O}_k, \gamma_i \in \psi(\mathcal{O}^1)\right\}.$$

Suppose that $\mathcal{D}$ is a maximal order of A for which $\psi(D)$ contains $\mathcal{O}\psi(\mathcal{O}^1)$. If $\mathcal{D} \neq \mathcal{O}$ then $[\Gamma_{\mathcal{O}} : P\psi(\mathcal{D}\cap\mathcal{O})^1] > 1$. But $\psi(\mathcal{D}\cap\mathcal{O})^1 \supset (\mathcal{O}\psi(\mathcal{O}^1))^1 \supset \psi(\mathcal{O}^1)$. Therefore $\mathcal{D} = \mathcal{O}$ and similarly, $\mathcal{O}'$ is the unique maximal order of A for which $\mathcal{O}\psi(\mathcal{O}'^1)$ is contained in $\psi(\mathcal{O}')$. As $\psi(a)$ conjugates $\mathcal{O}\psi(\mathcal{O}^1)$ to $\mathcal{O}\psi(\mathcal{O}'^1)$, a must conjugate $\mathcal{O}$ to $\mathcal{O}'$. □

8.3. **An example.** Let $k = \mathbb{Q}(\sqrt{-5})$ and consider the ideals $\mathfrak{p}_1 = (11)$ and $\mathfrak{p}_2 = (3+2\sqrt{-10})$ of $\mathbb{Q}(\sqrt{-5})$. These are both prime ideals and have norms 121 and 29 respectively. Let A be the quaternion division algebra over k defined by $\mathrm{Ram}(A) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$. In terms of Hilbert symbols A is given by $\left(\frac{44-11\sqrt{-5},-38-6\sqrt{-5}}{\mathbb{Q}(\sqrt{-5})}\right)$. All of this can be verified with the following Magma code.

```
> k<t>:=QuadraticField(-5);
> Zk:=Integers(k);
> p1:=11*Zk;
> IsPrime(p1);
true
> p2:=(3+2*t)*Zk;
> IsPrime(p2);
true
> Norm(p1);
121
> Norm(p2);
29
> A:=QuaternionAlgebra(p1*p2);
> Basis(A);
```

```
[ 1, i, j, k ]
> i:=Basis(A)[2];
> j:=Basis(A)[3];
> i^2;
(44-11*t)
> j^2;
(-38-6*t)
```

The prime ideal $\mathfrak{p}_1 = (11)$ splits completely in both $k(\sqrt{-1})$ and $k(\sqrt{-3})$, hence Theorem 7.4 implies that neither of these extensions embeds into A. No other cyclotomic extension of k is quadratic, hence A contains no roots of unity other than ± 1.

The type number of A is two, hence there exist maximal orders $\mathcal{O}$ and $\mathcal{O}'$ of A which are not conjugate.

```
> #ConjugacyClasses(MaximalOrder(A));
2
```

We have just shown that $\Gamma_{\mathcal{O}}$ and $\Gamma_{\mathcal{O}'}$ are torsion-free. It now follows from Theorem 8.8 and Proposition 8.11 that the arithmetic hyperbolic 3-manifolds $\Gamma_{\mathcal{O}}\backslash\mathbf{H}^3$ and $\Gamma_{\mathcal{O}'}\backslash\mathbf{H}^3$ are strongly isospectral but not isometric.

We now use Theorem 6.1 to compute the volume of our isospectral hyperbolic 3-manifolds. (Weyl's law implies that isospectral compact Riemmanian manifolds always have the same volume.) In this case we have

$$d_k = 20$$

and

$$\zeta_k(2) = 1.85555689374712063476271341165\dots$$

hence

$$\mathrm{Vol}(\Gamma_{\mathcal{O}}\backslash\mathbf{H}^3) = \mathrm{Vol}(\Gamma_{\mathcal{O}'}\backslash\mathbf{H}^3) = \frac{20^{3/2}\cdot(1.8555\dots)\cdot 120\cdot 28}{4\pi^2} = 14,125.336712\dots$$

Remark 8.12. *Note the Mostow's Rigidity Theorem implies that any isomorphism of $\Gamma_{\mathcal{O}}$ and $\Gamma_{\mathcal{O}'}$ would be induced by an isometry of $\Gamma_{\mathcal{O}}\backslash\mathbf{H}^3$ and $\Gamma_{\mathcal{O}'}\backslash\mathbf{H}^3$. It follows that our strongly isospectral non-isometric hyperbolic 3-manifolds have non-isomorphic fundamental groups.*

9. Type numbers and a theorem of Chinburg and Friedman

Let k be a number field and A be a quaternion algebra over k for which $A \otimes_{\mathbb{Q}} \mathbb{R} \not\cong \mathbb{H}^{[k:\mathbb{Q}]}$. In this section we will construct a Galois extension $k_{\mathcal{O}}$ of k with the property that A contains precisely $[k_{\mathcal{O}} : k]$ isomorphism classes of maximal orders. We will then use this extension in order to prove Theorem 8.7, a crucial ingredient in Section 8's construction of isospectral hyperbolic 3-manifolds.

9.1. **The local-global correspondence for orders.** Given a prime $\mathfrak{p}$ of k (possibly infinite) define $A_{\mathfrak{p}} := A \otimes_k k_{\mathfrak{p}}$.

Let $\mathcal{O}$ be a maximal order of A. We define the completion of $\mathcal{O}$ at a prime $\mathfrak{p}$ of k by

$$\mathcal{O}_{\mathfrak{p}} = \begin{cases} \mathcal{O} \otimes_{\mathcal{O}_k} \mathcal{O}_{k_{\mathfrak{p}}} & \text{if } \mathfrak{p} \text{ is finite} \\ \mathcal{O} \otimes_{\mathcal{O}_k} k_{\mathfrak{p}} = A_{\mathfrak{p}} & \text{if } \mathfrak{p} \text{ is infinite} \end{cases}$$

An important fact that we will use many times is that if $\mathcal{O}$ is a maximal order of A then $\mathcal{O}_{\mathfrak{p}}$ is a maximal order of $A_{\mathfrak{p}}$ for all finite primes $\mathfrak{p}$. For instance, if $A = \mathrm{M}_2(\mathbb{Q})$ then $\mathcal{O} = \mathrm{M}_2(\mathbb{Z})$ is a maximal order of A by Lemma 5.10. In this case $A_p = \mathrm{M}_2(\mathbb{Q}_p)$ and $\mathcal{O}_p = \mathrm{M}_2(\mathbb{Z}_p)$, the latter of which is a maximal order by Lemma 5.10.

Let $\mathcal{N}(\mathcal{O}_{\mathfrak{p}})$ denote the normalizer in $A_{\mathfrak{p}}^*$ of $\mathcal{O}_{\mathfrak{p}}$.

If $\mathfrak{p}$ is an infinite prime then $\mathcal{N}(\mathcal{O}_{\mathfrak{p}}) = A_{\mathfrak{p}}^*$ and $n(\mathcal{N}(\mathcal{O}_{\mathfrak{p}})) = k_{\mathfrak{p}}^*$.

When $\mathfrak{p}$ is a finite prime we have two cases to consider.

The first case is when $\mathfrak{p}$ ramifies in A. In this case $A_{\mathfrak{p}}$ is a division algebra and $\mathcal{O}_{\mathfrak{p}}$ is the unique maximal order of A. We thus have $\mathcal{N}(\mathcal{O}_{\mathfrak{p}}) = A_{\mathfrak{p}}^*$ and $n(\mathcal{N}(\mathcal{O}_{\mathfrak{p}})) = k_{\mathfrak{p}}^*$.

Now consider the case in which $\mathfrak{p}$ splits in A. Corollary 5.13 shows that in this situation we must have $\mathcal{O}_{\mathfrak{p}}$ conjugate to $\mathrm{M}_2(\mathcal{O}_{k_{\mathfrak{p}}})$, hence $\mathcal{N}(\mathcal{O}_{\mathfrak{p}})$ is conjugate to $\mathrm{GL}_2(\mathcal{O}_{k_{\mathfrak{p}}})k_{\mathfrak{p}}^*$ and $n(\mathcal{N}(\mathcal{O}_{\mathfrak{p}})) = \mathcal{O}_{k_{\mathfrak{p}}}^*(k_{\mathfrak{p}}^*)^2$.

We have just associated to every prime $\mathfrak{p}$ of k a maximal order $\mathcal{O}_{\mathfrak{p}}$ of $A_{\mathfrak{p}}$. The local-global correspondence for orders provides a converse to this association.

Theorem 9.1 (Local-Global Correspondence)**.** *Suppose that we are given an order $\mathcal{O}(\mathfrak{p})$ of $A_{\mathfrak{p}}$ for every finite prime $\mathfrak{p}$ of k. If there exists an order $\mathcal{O}$ of A such that $\mathcal{O}_{\mathfrak{p}} = \mathcal{O}(\mathfrak{p})$ for all but finitely many primes then there exists a unique order $\mathcal{O}$ of A for which $\mathcal{O}_{\mathfrak{p}} = \mathcal{O}(\mathfrak{p})$ for all finite primes $\mathfrak{p}$ of k. This order is given by*

$$\mathcal{O} = \bigcap_{\mathfrak{p}} A \cap \mathcal{O}(\mathfrak{p}).$$

For such an order $\mathcal{O}$ we have that $\mathcal{O}$ is a maximal order if and only if $\mathcal{O}(\mathfrak{p})$ if maximal for all finite primes $\mathfrak{p}$ of k.

9.2. **Type numbers and the idele group of a quaternion algebra.** We define the idele group J_A of A as follows:

$$J_A = \{(x_{\mathfrak{p}}) \in \prod_{\mathfrak{p}} A_{\mathfrak{p}}^* : x_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}^* \text{ for almost all finite primes } \mathfrak{p}\}.$$

Notice that we have a map $A^* \hookrightarrow J_A$ given by $y \mapsto (y, y, \dots)$. This map is called the *diagonal embedding* and allows us to view A^* as a subgroup of J_A. Note also that because any two orders of A are locally *equal* at all but finitely many primes, the definition of J_A does not depend on the choice of maximal order $\mathcal{O}$.

Given a maximal order $\mathcal{O}$ and idele $\tilde{x} = (x_\mathfrak{p}) \in J_A$ we define another maximal order $\tilde{x}\mathcal{O}\tilde{x}^{-1}$ (via the local-global correspondence) as being the unique maximal order of A for which

$$(\tilde{x}\mathcal{O}\tilde{x}^{-1})_\mathfrak{p} = x_\mathfrak{p}\mathcal{O}_\mathfrak{p}x_\mathfrak{p}^{-1}$$

holds for all finite primes $\mathfrak{p}$. As this notation suggests, if $\tilde{x} \in A^* \subset J_A$ then $\tilde{x}\mathcal{O}\tilde{x}^{-1}$ is *globally conjugate* to $\mathcal{O}$ by the element x; that is, $\tilde{x}\mathcal{O}\tilde{x}^{-1} = x\mathcal{O}x^{-1}$.

Lemma 9.2. *If $\mathcal{O}'$ is a maximal order of A then $\mathcal{O}' = \tilde{x}\mathcal{O}\tilde{x}^{-1}$ for some $\tilde{x} \in J_A$.*

Proof. Let S be the set of primes $\mathfrak{p}$ for which $\mathcal{O}'_\mathfrak{p} \neq \mathcal{O}_\mathfrak{p}$. This set is finite and satisfies $S \cap \mathrm{Ram}(A) = \emptyset$. Given a prime $\mathfrak{p} \in S$, note that $A_\mathfrak{p} \cong \mathrm{M}_2(k_\mathfrak{p})$ and that all maximal orders of $A_\mathfrak{p}$ are conjugate. Let $a_\mathfrak{p}$ be such that $\mathcal{O}'_\mathfrak{p} = a_\mathfrak{p}\mathcal{O}_\mathfrak{p}a_\mathfrak{p}^{-1}$. Now define an idele $\tilde{x} = (x_\mathfrak{p}) \in J_A$ by

$$x_\mathfrak{p} = \begin{cases} 1 & \text{if } \mathfrak{p} \notin S \\ a_\mathfrak{p} & \text{if } \mathfrak{p} \in S. \end{cases}$$

We now have $\mathcal{O}' = \tilde{x}\mathcal{O}\tilde{x}^{-1}$ by the uniqueness of the local-global correspondence, as both orders are locally equal at all finite primes of k. □

We have now defined a transitive action of J_A on the set of maximal orders of A by conjugation. The orbit of $\mathcal{O}$ under the subgroup A^* of J_A is simply the conjugacy class of $\mathcal{O}$ in A^*, hence we have an induced action of $A^*\backslash J_A$ on the conjugacy classes of maximal orders of A. The stabilizer of the conjugacy class of $\mathcal{O}$ under this action is

$$\mathcal{N}(\mathcal{O}) = J_A \cap \prod_\mathfrak{p} \mathcal{N}(\mathcal{O}_\mathfrak{p}).$$

We summarize this discussion as a proposition.

Proposition 9.3. *The set of conjugacy classes of maximal orders of A is in one to one correspondence with the double coset space $A^*\backslash J_A/\mathcal{N}(\mathcal{O})$. Under this correspondence two maximal orders $\mathcal{O}$ and $\mathcal{O}'$ are conjugate if and only if there is an idele $\tilde{x} \in J_A$ such that $\mathcal{O} = \tilde{x}\mathcal{O}'\tilde{x}^{-1}$ and $A^*\tilde{x}\mathcal{N}(\mathcal{O}) = A^*\mathcal{N}(\mathcal{O})$.*

Another way of stating Proposition 9.3 is that the type number of A is the cardinality of $A^*\backslash J_A/\mathcal{N}(\mathcal{O})$. Observe that we may extend the reduced norm of A to $A^*\backslash J_A/\mathcal{N}(\mathcal{O})$ as follows: $n(A^*\tilde{x}\mathcal{N}(\mathcal{O})) = k^* \cdot (n(x_\mathfrak{p})) \cdot n(\mathcal{N}(\mathcal{O}))$. This gives us a map

$$n : A^*\backslash J_A/\mathcal{N}(\mathcal{O}) \longrightarrow k^*\backslash J_k/n(\mathcal{N}(\mathcal{O})) \tag{5}$$

where J_k denotes the idele group of k. A consequence of the Strong Approximation Theorem is that this map is in fact a bijection [22, Theorem 3.3].

Theorem 9.4. *Let k be a number field and A be a quaternion algebra over k for which $A \otimes_\mathbb{Q} \mathbb{R} \not\cong \mathbb{H}^{[k:\mathbb{Q}]}$. The reduced norm map*

$$n : A^*\backslash J_A/\mathcal{N}(\mathcal{O}) \longrightarrow k^*\backslash J_k/n(\mathcal{N}(\mathcal{O}))$$

is a bijection.

Let $H_{\mathcal{O}} = k^* n(\mathcal{N}(\mathcal{O}))$ and $G_{\mathcal{O}} = J_k/H_{\mathcal{O}}$. As J_k is abelian we have $G_{\mathcal{O}} \cong k^*\backslash J_k/n(\mathcal{N}(\mathcal{O}))$. Because $H_{\mathcal{O}}$ is an open subgroup of J_k with finite index there is, by class field theory, a class field $k_{\mathcal{O}}$ for which $\mathrm{Gal}(k_{\mathcal{O}}/k) \cong G_{\mathcal{O}}$. We will call $k_{\mathcal{O}}$ the **type class field** of $\mathcal{O}$. Therefore the type number of A is equal to $[k_{\mathcal{O}} : k]$. The following is a standard result of class field theory.

Proposition 9.5. *A prime $\mathfrak{p}$ of k (possibly infinite) is unramified in $k_{\mathcal{O}}/k$ if and only if $\mathcal{O}_{\mathfrak{p}}^* \subset H_{\mathcal{O}}$ and splits completely if and only if $k_{\mathfrak{p}}^* \subset H_{\mathcal{O}}$.*

Proposition 9.5 and our computation of local normalizers in the beginning of this section show that $k_{\mathcal{O}}/k$ is unramified outside of the real places in $\mathrm{Ram}(A)$ and that every finite prime of $\mathrm{Ram}(A)$ splits completely in $k_{\mathcal{O}}/k$. Because $J_k \subset \mathcal{N}(\mathcal{O})$, hence $J_k^2 \subset n(\mathcal{N}(\mathcal{O}))$, the group $G_{\mathcal{O}}$ has exponent 2. Putting all of this together we have that $k_{\mathcal{O}}$ is the maximal abelian extension of k which has exponent 2, is unramified outside of the real places in $\mathrm{Ram}(A)$ and in which every finite prime of $\mathrm{Ram}(A)$ splits completely.

Corollary 9.6. *Let k be a number field and A be a quaternion algebra over k for which $A \otimes_{\mathbb{Q}} \mathbb{R} \ncong \mathbb{H}^{[k:\mathbb{Q}]}$. The type number of A is 2^t for some $t \geq 0$.*

Proof. We have seen that the type number of A is equal to the cardinality of $G_{\mathcal{O}}$ and have just shown that the latter group is a finite abelian group of exponent 2. It follows that $G_{\mathcal{O}} \cong (\mathbb{Z}/2\mathbb{Z})^t$ for some $t \geq 0$. The result follows. □

9.3. **A more refined analysis of type class fields.** In order to prove Theorem 8.7 we will need a more refined analysis of the type class field $k_{\mathcal{O}}$. We begin by proving two lemmas which clarify the extend to which we can control the generators of the group $G_{\mathcal{O}}$.

Lemma 9.7. *The group $G_{\mathcal{O}} = J_{k/H_{\mathcal{O}}}$ is generated by cosets having representatives of the form $e_{\mathfrak{p}_i} = (1, \ldots, 1, \pi_{\mathfrak{p}_i}, 1, \ldots)$. If S is any finite set of primes of k then the elements $\{e_{\mathfrak{p}_i}\}$ can be chosen so that $\mathfrak{p}_i \notin S$ for all i.*

Proof. The Chebotarev Density Theorem shows that every element of $\mathrm{Gal}(k_{\mathcal{O}}/k)$ has infinitely many prime ideals in its preimage under the Artin map. As these prime ideals correspond to ideles of the form $e_{\mathfrak{p}_i} = (1, \ldots, 1, \pi_{\mathfrak{p}_i}, 1, \ldots)$, $G_{\mathcal{O}}$ can be represented by cosets having the correct form. Because each element of $G_{\mathcal{O}}$ has infinitely many such preimages and S is finite, we may take $\mathfrak{p}_i \notin S$ for all i. □

Lemma 9.8. *Let L be a quadratic field extension of k and suppose that $L \not\subset k_{\mathcal{O}}$. Then $G_{\mathcal{O}}$ is generated by cosets $e_{\mathfrak{p}_i} H_{\mathcal{O}}$ where $\mathfrak{p}_i$ splits in L/k for all i.*

Proof. By the Chebotarev density theorem we may generate $\mathrm{Gal}(k_{\mathcal{O}}L/L)$ with the Frobenius elements associated to primes of L having degree one over k (since the set of primes of L with degree greater than one over k has density zero). As $\mathrm{Gal}(k_{\mathcal{O}}L/L)$ is isomorphic to $\mathrm{Gal}(k_{\mathcal{O}}/k)$ via the map $\sigma \mapsto \sigma|_{k_{\mathcal{O}}}$, we may generate the latter group with Frobenius elements associated to primes of k splitting completely in L/k. These automorphisms correspond, via the Artin map, to the generators $e_{\mathfrak{p}_i} H_{\mathcal{O}}$ of $G_{\mathcal{O}}$. □

Theorem 9.9 (Chinburg-Friedman)**.** *Let k be a number field and A a quaternion algebra over k for which $A \otimes_{\mathbb{Q}} \mathbb{R} \ncong \mathbb{H}^{[k:\mathbb{Q}]}$. Let L be a quadratic field extension of k and $\Omega \subset L$ a quadratic $\mathcal{O}_k$-order. If A is ramified at a finite prime of k and Ω embeds into a maximal order of A then every maximal order of A admits an embedding of Ω.*

Proof. We have two cases to consider. Suppose first that $L \not\subset k_{\mathcal{O}}$ and $L \subset A$. We may assume, without loss of generality, that $\Omega \subset \mathcal{O}$. By Lemmas 9.7 and 9.8 we may generate $G_{\mathcal{O}}$ with coset representatives $e_{\mathfrak{p}_i} H_{\mathcal{O}}$ with all of the primes $\mathfrak{p}_i$ splitting in L/k. Let $\mathfrak{p}$ be one such prime $\mathfrak{p}_i$. Because $\mathfrak{p}$ splits in L/k, Theorem 7.4 implies that $\mathfrak{p}$ splits in A and there is a $k_{\mathfrak{p}}$-isomorphism

$$f_{\mathfrak{p}} : A_{\mathfrak{p}} \to \mathrm{M}_2(k_{\mathfrak{p}})$$

for which

$$f_{\mathfrak{p}}(L) \subset \begin{pmatrix} k_{\mathfrak{p}} & 0 \\ 0 & k_{\mathfrak{p}} \end{pmatrix}.$$

Consequently

$$f_{\mathfrak{p}}(\Omega) \subset \begin{pmatrix} \mathcal{O}_{k_{\mathfrak{p}}} & 0 \\ 0 & \mathcal{O}_{k_{\mathfrak{p}}} \end{pmatrix}.$$

Define two orders $\mathcal{D}_{\mathfrak{p}}$ and $\mathcal{D}'_{\mathfrak{p}}$ of $A_{\mathfrak{p}}$ by

$$\mathcal{D}_{\mathfrak{p}} = f_{\mathfrak{p}}^{-1}\left(\begin{pmatrix} \mathcal{O}_{k_{\mathfrak{p}}} & \mathcal{O}_{k_{\mathfrak{p}}} \\ \mathcal{O}_{k_{\mathfrak{p}}} & \mathcal{O}_{k_{\mathfrak{p}}} \end{pmatrix}\right), \qquad \mathcal{D}'_{\mathfrak{p}} = f_{\mathfrak{p}}^{-1}\left(\begin{pmatrix} \mathcal{O}_{k_{\mathfrak{p}}} & \pi_{\mathfrak{p}}^{-1}\mathcal{O}_{k_{\mathfrak{p}}} \\ \pi_{\mathfrak{p}}\mathcal{O}_{k_{\mathfrak{p}}} & \mathcal{O}_{k_{\mathfrak{p}}} \end{pmatrix}\right)$$

and observe that $\mathcal{D}_{\mathfrak{p}}$ and $\mathcal{D}'_{\mathfrak{p}}$ are conjugate by the matrix $f_{\mathfrak{p}}^{-1}\left(\begin{pmatrix} \pi_{k_{\mathfrak{p}}} & 0 \\ 0 & 1 \end{pmatrix}\right) \in \mathrm{GL}_2(k_{\mathfrak{p}})$. Recall that $G_{\mathcal{O}} \cong (\mathbb{Z}/2\mathbb{Z})^t$. Given an element $\gamma = (\gamma_i) \in (\mathbb{Z}/2\mathbb{Z})^t$, define an order $\mathcal{D}^{\gamma}$ of A by

$$\mathcal{D}^{\gamma}_{\mathfrak{p}} = \begin{cases} \mathcal{D}_{\mathfrak{p}_i} & \text{if } \mathfrak{p} = \mathfrak{p}_i \text{ and } \gamma_i = 0 \\ \mathcal{D}'_{\mathfrak{p}_i} & \text{if } \mathfrak{p} = \mathfrak{p}_i \text{ and } \gamma_i = 1 \\ \mathcal{O}_{\mathfrak{p}_i} & \text{otherwise.} \end{cases}$$

Such an order exists by the local-global correspondence for orders. We have therefore defined 2^t maximal orders of A. We claim that these orders represent all 2^t isomorphism classes of maximal orders of A. Indeed, $\mathcal{D}^{\gamma}$ and $\mathcal{D}^{\gamma'}$ are conjugate if and only if the idele $\tilde{x} \in J_A$ for which $\mathcal{D}^{\gamma} = \tilde{x}\mathcal{D}^{\gamma'}\tilde{x}^{-1}$ satisfies $A^*\tilde{x}\mathcal{N}(\mathcal{O}) = A^*\mathcal{N}(\mathcal{O})$ in the double coset space $A^*\backslash J_A/\mathcal{N}(\mathcal{O})$. If $\gamma \neq \gamma'$ then the claim follows from Theorem 9.4 because

$$n\left(\begin{pmatrix} \pi_{k_{\mathfrak{p}}} & 0 \\ 0 & 1 \end{pmatrix}\right) = \pi_{k_{\mathfrak{p}}}$$

and the $\mathfrak{p}_i$ were chosen so that $e_{\mathfrak{p}_i} = (1, \dots, 1, \pi_{k_{\mathfrak{p}_i}}, 1, \dots)$ is nontrivial in $G_{\mathcal{O}} \cong k^*\backslash J_k/n(\mathcal{N}(\mathcal{O}))$. This proves that every maximal order of A is conjugate to one of the $\mathcal{D}^{\gamma}$. Because Ω is contained in all of the $\mathcal{D}^{\gamma}$ locally at all finite primes, the local-global correspondence implies that Ω is contained in $\mathcal{D}^{\gamma}$ for all $\gamma = (\gamma_i) \in (\mathbb{Z}/2\mathbb{Z})^t$. In particular every maximal order of A admits an embedding of Ω.

Suppose now that $L \subset k_{\mathcal{O}}$ and let $\mathfrak{p}$ be a finite prime of k which ramifies in A. We have already seen that such a prime splits completely in $k_{\mathcal{O}}/k$. As L is a subfield of $k_{\mathcal{O}}$, the prime $\mathfrak{p}$ splits in L/k as well. Theorem 7.4 thus implies that L does not embed into A, hence no maximal order of A admits an embedding of Ω. $\square$

10. Sizes of isospectral families of arithmetic hyperbolic 3-manifolds

In Sections 8 and 9 we constructed arithmetic hyperbolic 3-manifolds and developed the number theory necessary to prove that their isospectrality. Recall that our examples were of the form $\{\mathbf{H}^3/\Gamma_{\mathcal{O}}, \mathbf{H}^3/\Gamma_{\mathcal{O}'}\}$ where $\mathcal{O}$ and $\mathcal{O}'$ are maximal orders in a suitable quaternion algebra A. Because Vignéras was the first to construct isospectral hyperbolic manifolds in this manner [35], we will say that $\{\mathbf{H}^3/\Gamma_{\mathcal{O}}, \mathbf{H}^3/\Gamma_{\mathcal{O}'}\}$ arise via **Vignéras' method**. In order to ensure that our examples were not isometric, we took $\mathcal{O}$ and $\mathcal{O}'$ to be non-conjugate in A. It follows that a family of (pairwise) isospectral non-isometric hyperbolic 3-manifolds constructed via Vignéras' method has cardinality bounded above by the type number of A. In this section we will use the formula for the volume of arithmetic Kleinian groups of the simplest type in order to prove that a family of isospectral non-isometric hyperbolic 3-manifolds of volume V constructed via Vignéras' method has cardinality at most cV^2 for some positive constant c. Our proof will follow [23].

Theorem 10.1. *The cardinality of a family of pairwise isospectral non-isometric hyperbolic 3-manifolds of volume V constructed via Vignéras' method is cV^2 for some absolute constant $c > 0$.*

The remainder of this section will be devoted to the proof of Theorem 10.1.

Consider a family $\mathbf{H}^3/\Gamma_{\mathcal{O}_1}, \dots, \mathbf{H}^3/\Gamma_{\mathcal{O}_n}$ of isospectral non-isometric hyperbolic 3-manifolds of volume V arising from Vignéras' method. Here $\mathcal{O}_1, \dots, \mathcal{O}_n$ are pairwise non-conjugate maximal orders in a quaternion algebra A defined over a number field k where k has a unique complex place and A is ramified at all real places of k.

Recall that in Theorem 6.1 we saw that the volume of the manifolds $\mathbf{H}^3/\Gamma_{\mathcal{O}_i}$ satisfy

$$V = \mathrm{Vol}(\mathbf{H}^3/\Gamma_{\mathcal{O}_i}) = \frac{d_k^{3/2}\zeta_k(2)}{(4\pi^2)^{n_k-2}} \cdot \left(\prod_{\mathfrak{p} \in \mathrm{Ram}_f(A)} (N(\mathfrak{p}) - 1) \right),$$

where $n_k = [k : \mathbb{Q}]$ and $\zeta_k(2)$ is the Dedekind zeta function of k evaluated at $s = 2$. Employing the trivial bounds $\zeta_k(2) \geq 1$ and $N(\mathfrak{p}) \geq 2$ for all primes $\mathfrak{p}$ of k, we obtain the inequality

$$V \geq d_k^{3/2}/(4\pi^2)^{n_k-2}. \tag{6}$$

In Section 9 we proved the existence of an abelian extension $k_{\mathcal{O}}/k$ whose degree is equal to the type number of A. We moreover saw that $k_{\mathcal{O}}$ is unramified outside of the real places in $\mathrm{Ram}(A)$. In particular $k_{\mathcal{O}}$ is contained in the narrow class field of k and hence has degree over k at most $h_k 2^{n_k-2}$ where h_k is the class number of k. In order to proceed we will require the following class number estimate [23, Lemma 3.1]:

Lemma 10.2. *Let k be a number field with a unique complex place, degree n_k, class number h_k and absolute value of discriminant d_k. Then*

$$h_k \leq \frac{242 d_k^{3/4}}{(1.64)^{n_k-2}}.$$

In light of Lemma 10.2 and the previous paragraph we see that the cardinality of a family of isospectral non-isometric hyperbolic 3-manifolds of volume V arising from

Vignéras' method is at most $242(1.22)^{n_k-2}d_k^{3/4}$. We will now simplify this expression, as well as (6), by bounding n_k in terms of d_k. We will do so by employing the Odlyzko discriminant bounds [28].

Theorem 10.3 (Odlyzko)**.** *Let k be a number field of signature (r_1, r_2), degree $n_k = r_1 + 2r_2$ and absolute value of discriminant d_k. There is an absolute constant $C > 0$ such that*

$$\log(d_k) \geq r_1 + n_k(\gamma + \log(4\pi)) - C,$$

where $\gamma = 0.57721\ldots$ is the Euler-Mascheroni constant.

In the case of interest to us the number field k has signature $(n_k - 2, 1)$, hence Theorem 10.3 shows that there is an absolute constant $C > 0$ such that

$$n_k \leq \log(d_k^{1/4}) + C. \tag{7}$$

Using equations (6) and (7) we see that

$$V \geq d_k^{3/2}/(4\pi^2)^{n_k-2} \geq d_k^{3/2}/e^{4n_k-8} \geq C' d_k^{1/2} \tag{8}$$

for some absolute constant C'.

Using equation (7) we deduce that the cardinality of a family of isospectral non-isometric hyperbolic 3-manifolds of volume V arising from Vignéras' method is at most

$$242(1.22)^{n_k-2}d_k^{3/4} \leq 163 d_k.$$

Theorem 10.1 now follows from equation (8).

11. Contrasting the methods of Vignéras and Sunada

In Section 8 we constructed isospectral arithmetic hyperbolic 3-manifolds using what is now known as *Vignéras' method*. In this section we will introduce a different method of constructing isospectral Riemannian manifolds: *Sunada's method*. Sunada's method reduces the problem of constructing isospectral manifolds to a problem in finite group theory. Because of the elementary nature of Sunada's method, the majority of the known examples of isospectral Riemannian manifolds have been constructed using it and its generalizations. We will conclude this section by showing that the methods of Vignéras and Sunada are incompatible. In other words, isospectral hyperbolic 3-manifolds constructed using Vignéras' method can never arise via Sunada's method.

11.1. **Sunada's method.** As was mentioned above, the idea behind Sunada's method is to reduce the construction of isospectral Riemannian manifolds to a problem in finite group theory. We begin with a definition.

Let G be a finite group and $g \in G$. We denote by $[g]$ the G-conjugacy class of g. We say that two subgroups H_1, H_2 of G are **almost conjugate** if

$$\#(H_1 \cap [g]) = \#(H_2 \cap [g])$$

for all $g \in G$.

Example 11.1. In the group $G = S_6$, a pair of nonconjugate almost-conjugate subgroups are given by

$$H_1 = \{(1), (12)(34), (13)(24), (14)(23)\}$$

and

$$H_2 = \{(1), (12)(34), (12)(56), (34)(56)\}.$$

Theorem 11.2 (Sunada [34]). *Let M be a closed Riemannian manifold, G a finite group and*

$$\varphi : \pi_1(M) \longrightarrow G$$

a surjective homomorphism. If H_1 and H_2 are almost-conjugate subgroups of G then the manifolds $M/\varphi^{-1}(H_1)$ and $M/\varphi^{-1}(H_2)$ are isospectral.

Remark 11.3. *Sunada's method actually produces manifolds that are not only isospectral but which are in fact strongly isospectral.*

Observe that isospectral manifolds arising via Sunada's method are always commensurable; that is, they always have a common, finite degree covering space: namely, M. They furthermore always have a common, finite degree quotient manifold: namely, M/G. As the latter property is not satisfied by isospectral manifolds arising via Vignéras' method and can therefore be used to show that the two methods are incompatible, we record these observations as a corollary.

Corollary 11.4. *Isospectral manifolds arising via Sunada's method are always commensurable and always have a common, finite degree quotient manifold.*

Before proving that the methods of Vignéras and Sunada are incompatible we will apply Sunada's method so as to construct isospectral hyperbolic 3-manifolds. To that end, let M be a closed hyperbolic 3-manifold. A celebrated theorem of Agol [1, Theorem 9.2] shows that $\pi_1(M)$ is *large*; that is, $\pi_1(M)$ surjects onto a non-abelian free group. In particular $\pi_1(M)$ surjects onto the free group on two generators and hence onto any finite group generated by two elements. As the group S_6 is generated by two elements, we have a surjection

$$\varphi : \pi_1(M) \to S_6.$$

Let H_1 and H_2 be the subgroups of S_6 given in Example 11.1. We conclude from Theorem 11.2 that the quotient manifolds $M/\varphi^{-1}(H_1)$ and $M/\varphi^{-1}(H_2)$ are isospectral.

11.2. **Vignéras' examples cannot arise from Sunada's method.** We now show that the methods of Vignéras and Sunada are incompatible. Our proof will follow that of Chen [6].

We begin with a lemma.

Lemma 11.5. *Let k be a number field, A a quaternion algebra over k and $\mathcal{O} \subset A$ a maximal order. If $\mathcal{L}$ is a subgroup of A^1 containing $\mathcal{O}^1$ and $[\mathcal{L} : \mathcal{O}^1] < \infty$ then $\mathcal{L} = \mathcal{O}^1$.*

Proof. Let $L = \mathcal{O}_k[\mathcal{L}]$ be the ring generated over $\mathcal{O}_k$ by $\mathcal{L}$. Since $[\mathcal{L} : \mathcal{O}^1] < \infty$ we may write $\mathcal{L} = \bigcup g_i\mathcal{O}^1$, hence $L = \sum \mathcal{O}_k\{g_i\mathcal{O}^1\}$ is a finitely generated $\mathcal{O}_k$-module containing $\mathcal{O}_k[\mathcal{O}^1]$; that is, L is an order of A. Since L^1 contains $\mathcal{L}$, which in turn contains $\mathcal{O}^1$, we conclude, by maximality of $\mathcal{O}$, that $\mathcal{L} = \mathcal{O}^1 = L$. □

We now prove the section's main result.

Theorem 11.6. *Isospectral manifolds arising from Vignéras' method do not have any finite degree quotient manifolds in common (up to isometry). In particular isospectral manifolds arising from Vignéras' method never arise from Sunada's method.*

Proof. Let k be a number field with a unique complex place and A a quaternion algebra over k in which all real places of k ramify. Let $\mathcal{O}$ and $\mathcal{O}'$ be non-conjugate maximal orders of A for which $\Gamma_{\mathcal{O}}\backslash\mathbf{H}^3$ and $\Gamma_{\mathcal{O}'}\backslash\mathbf{H}^3$ are Vignéras-isospectral hyperbolic 3-manifolds. Let N and N' be hyperbolic 3-manifolds which are isometric to $\Gamma_{\mathcal{O}}\backslash\mathbf{H}^3$ and $\Gamma_{\mathcal{O}'}\backslash\mathbf{H}^3$ and have a common finite degree quotient manifold $\Gamma_0\backslash\mathbf{H}^3$. Upon conjugation we may assume that $\Gamma_{\mathcal{O}}, \gamma\Gamma_{\mathcal{O}'}\gamma^{-1} \subseteq \Gamma_0$ for some $\gamma \in \mathrm{PSL}_2(\mathbb{C})$. By Theorem 6.13 we have that $\gamma \in P(\psi(A^1))$, hence Lemma 11.5 implies that Γ_0 cannot contain $\Gamma_{\mathcal{O}}$ with finite index. This contradicts our assumption that $\Gamma_{\mathcal{O}}\backslash\mathbf{H}^3$ has $\Gamma_0\backslash\mathbf{H}^3$ as a finite degree quotient manifold, proving our result. □

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Department of Mathematics, 530 Church Street, University of Michigan, Ann Arbor, MI 48109